Coherent Structure generating Automata

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In this article, four classes of cellular automata are constructed. Each was constructed with the intent of having a relatively simple rule (i.e with as few states as possible, recognizing only one state as excited) that has the same (discretized) behaviour as the solutions of some partial differential equations. In each of these solutions, pulses are repeatedly generated by a (potentially) moving or reflecting source. The automata have each been constructed such that the pulses propagate exactly as for the one-dimensional Greenberg-Hastings Automaton[2] (with one excited and one refractory state). The four patterns observed by these solutions are the one-dimensional target pattern, the one-dimensional spiral, pulse-replication pattern and pulse splitting pattern.

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1 Introduction

The occurrence of self-similar and self-replicating patterns, which are quite common within the context of cellular automata, are of course not unique to them. These patterns have (among other fields) also been observed within the field of partial differential equations. For example, the formation of the Sierpinski gasket, a pattern which can be produced by the simple XOR automaton, has been shown[2] to appear in numerical simulations of several reaction-diffusion systems, when suitable parameters are chosen. However, it is also possible to investigate in the opposite direction. Partial differential equations can exhibit many self-replicating patterns, with this article focusing on those which produce a steady stream of pulses. Of these, four kinds will be examined.

In a pulse replication pattern, pulses are periodically generated from an invisible defect line, moving in the same spatial direction as the defect line itself. For the splitting pattern, the same is observed except that two pulses are generated every period, each moving in the opposite direction from the other. The one-dimensional target pattern is a special case of splitting, where the defect line does not move in space. Finally, the one-dimensional spiral is a variant of the one-dimensional target pattern, where only one pulse is generated every period, but the direction in which the new pulse propagates switches each time. The question arises whether a cellular automaton exists that can discretize these dynamics. The answer is of course yes, though it requires the use of automata with more than two states. Within the above phenomena, pulses (excitations) can be produced even if no excitation has happened in the vicinity, whether spatially or temporally, and yet excited states are not produced everywhere purely from rest states. This requires the use of 'intermediate' states which code certain states that the system is in.
This paper first introduces the Greenberg-Hastings automaton, whose behaviour for pulses all the other automata are based on. Then four automata are introduced which exhibit the behaviour of solutions of partial differential equations. Proofs of the correctness of the automata, due to their technical nature, have been moved to the appendix. Further, whether and how they annihilate is discussed. Finally, a lower bound on the topological entropy of these automata is given.

2 The Automata

Since these automata should be incapable of distinguishing between which of any cell's neighbours is the left or right, it will be useful for their definitions to have a shorthand for 'also include the reflected version of this neighborhood configuration':
$c : \mathcal{P}(\mathbb{Z}^3) \to \mathcal{P}(\mathbb{Z}^3)$
$A \mapsto A \cup \{(z, y, x) \in \mathbb{Z}^3 | \quad (x, y, z) \in A\}$

2.1 Greenberg-Hastings

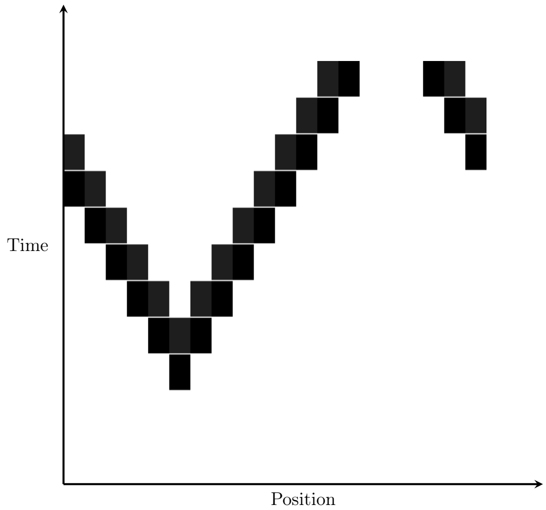

Figure 1: Evolution from a single excited cell with peridoic boundary condition

The one-dimensional Greenberg-Hastings automaton (GH) with one refractory state is the following:
$G : \{0,1,2\}^{\mathbb{Z}} \to \{0,1,2\}^{\mathbb{Z}}$

$$(a_n)_{n\in\mathbb{Z}} \mapsto \left(\begin{cases} 1, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(1,0,0)\}) \\ 2, & a_n = 1 \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$$

The idea behind this automaton is the following:
The excited state (1) excites its two neighbouring cells in the next time step, provided those cells are at rest (0). This way, excitation propagates. However, rather than an excited state immediately transitioning into the rest state, it first enters the so-called refractory state (2), whose only purpose, essentially, is to be 'inert'. It can't be excited, as it always transitions into the rest state. The most notable consequence of this is that the pattern of 0 1 2 self-replicates towards the left. This can be seen in Figure 1. At first, the system starts of with one excited cell (black) with the rest at rest (white). This allows the excited cell to propagate in both directions, left and right, in the next time step. However, at this point the cells in the middle are 0 1 2 1 0 (2 being grey). The refractory state blocks the excitation of its cell in the next step, meaning both excited cells can only excite one cell, specifically in the direction away from the middle. This repeats endlessly, the two created pulses moving away from each other, creating the two diagonals in the space-time grid. Generating these pulses, or waves, is the phenomenon that all the following automata were constructed for, with the intent that the pulses themselves behave just like they do for GH.

2.2 Pulse Replication

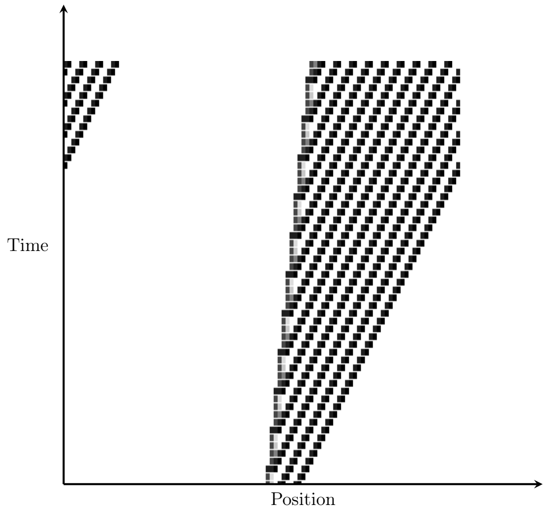

Figure 2: Pulse Replication initialised with a single defect line

The pulse replication pattern has been observed for the oregnotator process[6]. In pulse replication, pulses are continuosly created in a single direction along a defect line, which moves in the same direction.
When viewing this through the lens of the space-time grid, the defect line is an invisible trajectory along which pulses are continuously created, such that they all move in the same spatial direction as the defect line itself, although at a greater speed; the slope of the defect line (in space-time) is steeper.

2.2.1 The Construction

To encode this kind of behaviour into a cellular automaton, states beside 0,1,2 are required to form the defect line.
Given a height and width of a line segment between the two closest pulses, $(h, w) \in \mathbb{N}^2_{>0}$ such that $u := h - w + 1 \geqslant 4, h \geqslant 4$, such a cellular automaton exists:
Let $F_{h,w} : \{0, \cdots, h+1\}^{\mathbb{Z}} \to \{0, \cdots, h+1\}^{\mathbb{Z}}$
$(a_n)_{n\in\mathbb{Z}} \mapsto$

$$\left(\begin{cases} 1, & (a_{n-1}, a_n, a_{n+1}) \in c(\{1, h+1\} \times \{0\} \times \{0, 2\}) \\ 2, & a_n = 1 \vee (a_{n-1}, a_n, a_{n+1}) \in c(\{(0, 3, h+1), (0, h+1, 3)\}) \\ 3, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(2, 2, 1)\} \uplus \{0\} \times \{3\} \times \{2, \cdots, u\} \uplus \{3\} \times \{u+1, \cdots, h\} \times \{0\}) \\ 4, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(3, 2, 1)\}) \\ a_n + 1 & 4 \leqslant a_n \leqslant u \\ max\{a_{n-1}, a_n\} + 1, & (a_{n-1}, a_n, a_{n+1}) \in c(\{0, 2\} \times \{0\} \times \{u+1, \cdots, h\}) \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$$

Then, given the initial state: $2, 2, 1$ (with positions 0,1,2), excited states will be generated along the vector (h, m), which in turn generate excited states diagonally in the spatial direction of the vector (h, m).
The states from 4 to $u + 1$ are the states which are the purely vertical component of the vector, which simply count up, and remain in the same cell. The states from $u + 2$ to $h + 1$ are the diagonal component and increment to either the left or right (depending on the initial conditions). The state 3 is responsible for a variety of tasks. It is necessary to generate the first of the states that climb vertically, 4, as well as to generate a shifted copy of the initial state, and finally and most crucially, the 3 state encodes the direction of the vector. It does this by replicating itself each step along the vertical component and the diagonal component on the side opposite to the (spatial) direction of the vector, blocking movement into its direction.

2.2.2 Annihilation

Given $j \in \mathbb{N}_0$ an initial state $\left(\begin{cases} 1, & |n| = j \\ 2, & |n| \in \{j+1, j+2\} \\ 0, & \text{otherwise} \end{cases}\right)_{n \in \mathbb{Z}}$, iterating $F_{h,m}$ on it will eventually yield the zero function.

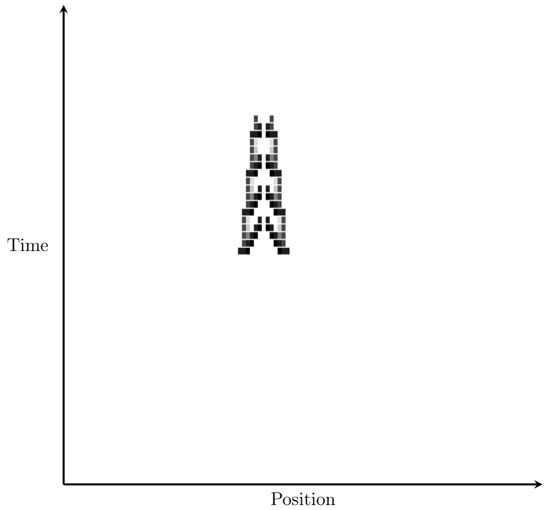

Figure 3: Two inward-facing initial states annihilating

2.2.3 Correctness

It turns out (is proven in the appendix) that a cell at position $n \in \mathbb{N}$ at time $m \in \mathbb{N}$ (that is, after m iterations of $F_{h,w}$) is excited if and only if $\exists k \in \mathbb{N} : \text{wave}(n, m, k)$

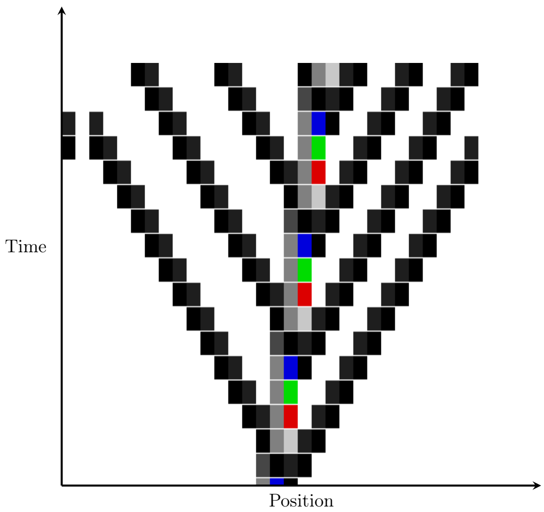

Figure 4: Splitting with one initial state ($h = 5, w = 1$)

With $\text{wave}(n, m, k) \equiv m - k \cdot h = n - 2 - k \cdot w \geqslant 0$
The k is the number of times the pattern has fully self-replicated before the pattern that created the wave that this excited state is on. Put another way, the excited state is a part of the $k + 1$th wave.
From this, we can easily see that $\text{wave}(n, m, k) \to \text{wave}(n + 1, m + 1, k)$, that is, each wave moves as for GH.
Additionally, it can also easily be seen that $\text{wave}(n, m, k) \to \text{wave}(n + w, m + h, k + 1)$
Notably, if we apply this to the first excited state of a new wave, then we can see that a new wave is generated after h more time passes at a position w more cells away, as required of the automaton.

2.3 Pulse Splitting

Splitting is the same as pulse replication, except that pulses are created in both spatial directions. It can be observed in the Gray Scott model[1].

2.3.1 The Construction

As for pulse replication, let the vector $(h, w) \in \mathbb{N}^2_{>0}$ describe the line segment between two closest pulses. Under the conditions that $h \geqslant 5, h - w \geqslant 3$, the automaton exists:

$$P_{h,w} : \{0, \cdots, h+3\}^{\mathbb{Z}} \to \{0, \cdots, h+3\}^{\mathbb{Z}}$$
$$(a_n)_{n\in\mathbb{Z}} \mapsto \left(\begin{cases} 1, & a_n = h+3 \vee a_n = 3 \vee (a_{n-1}, a_n, a_{n+1}) \in c(\{(1,0,0), (1,0,2), (0,0,h+3-1)\}) \\ 2, & a_n = 1 \wedge (a_{n-1}, a_n, a_{n+1}) \notin c(\{(3,1,2)\}) \\ 5, & (a_{n-1}, a_n, a_{n+1}) = (1,2,1) \\ a_n + 1, & (5 \leqslant a_n < j) \vee a_n = h+3-1 \\ max\{a_{n-1}, a_n\} + 1, & (a_{n-1}, a_n, a_{n+1}) \in c(\{0\}^2 \times \{h-w+4, \cdots, h+3-2\}) \\ 3, & (a_{n-1}, a_n, a_n) \in c(\{(h+3, 4, 0)\}) \\ 4, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(3,1,2)\} \uplus \mathbb{N}_0 \times \{4\} \times \{5, \cdots, h-w+3, h+2\} \uplus \{4\} \times \{h-w+4, \cdots, h+1\} \times \{0\}) \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$$

Given an initial state of $4, h+3, 1$, the automaton has the desired behaviour.
The above rule works similarly to Pulse Replication in terms of how the generation of new diagonal lines along a vector is realised by a vertical and a diagonal component. In this case, 4 is the state with the same function as 3 for Pulse Replication. There are two main differences. Firstly, the excited states along the vector are allowed to branch in both directions. Secondly, the state 3 is used to replace a 2 on the opposite side of the direction of the vector with a 4, which in turn encodes the direction of the vector. Without overwriting a 2 (or technically a 1, although this would break the desired pattern), asymmetry could not be introduced to the middle of the pattern.

2.3.2 Correctness

A formal proof of its correctness has been omitted but would essentially involve making similar arguments to those for the proof of correctness of pulse replication.

2.3.3 Annihilation

Given an initial state $(a_n)_{n\in\mathbb{Z}} \left(\begin{cases} 1, & |n| = h-w+4 \\ h+3, & |n| = h-w+4+1 \\ 4, & |n| = h-w+4+2 \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$, the sequence $(P^i_{h,w}((a_n)_{n\in\mathbb{Z}}))_{i\in\mathbb{N}}$ converges pointwise against the zero function (that is, each point permanently becomes 0 in finite time).

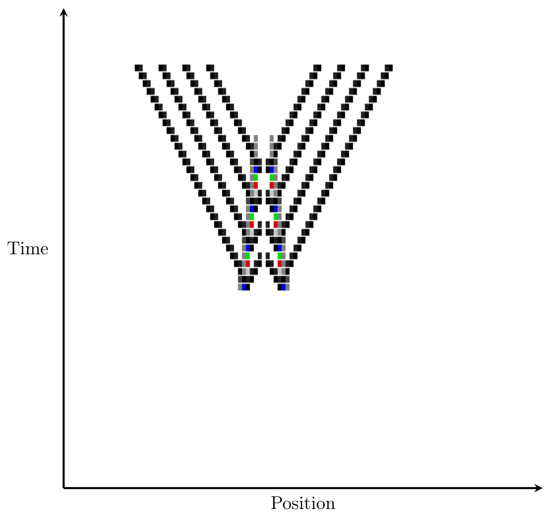

Figure 5: Two inward-facing initial states annihilating

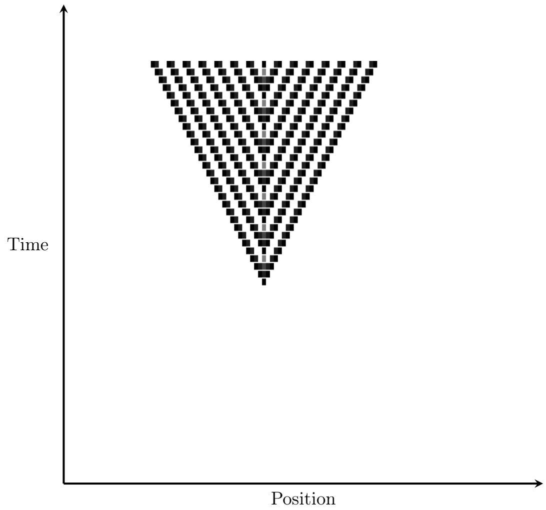

Figure 6: 1D Target Pattern

2.4 1D Target Pattern

The one-dimensional target pattern is the simplest of all of these patterns. A fixed point in space periodically produces pulses in both directions, creating a tower of V's in the space time grid. It is essentially the splitting pattern with a vertical defect line. It can be observed in some solutions to the Belousov-Zhabotinskii reaction[5].

2.4.1 The Construction

Let $h \in \mathbb{N}_{>2}$ be the period.
Let $T_h : \{0, \cdots, h\}^{\mathbb{Z}} \to \{0, \cdots, h\}^{\mathbb{Z}}$

$$(a_n)_{n\in\mathbb{Z}} \mapsto \left(\begin{cases} 1, & a_n = h \vee (a_{n-1}, a_n, a_{n+1}) \in c(\{(1,0,0),(1,0,2)\}) \\ 2, & a_n = 1 \\ 3, & (a_{n-1}, a_n, a_{n+1}) = (1,2,1) \\ a_n + 1 & 3 \leqslant a_n < h \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$$

This rule is fairly straightforward. It generates the starting point of a V-shape every h cells.

2.4.2 Correctness

The correctness can be readily read off of the construction by noting that only an excited cell that generates pulses (that is, the only one where it and its neighbours are followed by 1 2 1) creates the remaining states $3, \cdots, h$ which count up to return to excitation.

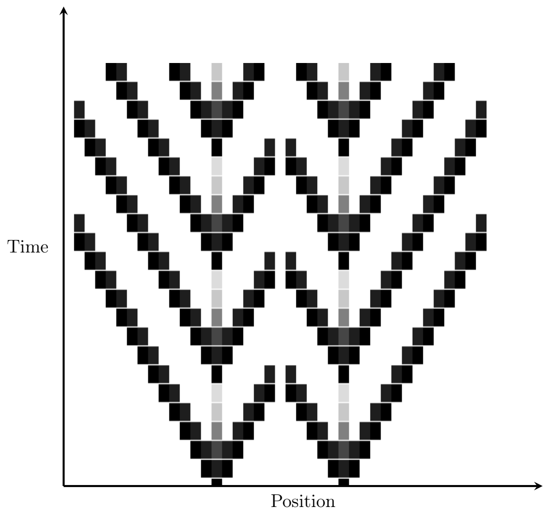

Figure 7: Two one-dimensional target patterns next to each other

2.4.3 Annihilation

Since the defect lines do not move, they do not annihilate. Instead, the pulses they send in each other's direction annihilate, with the pulses sent out in the opposite directions unaffected.

1D Spiral

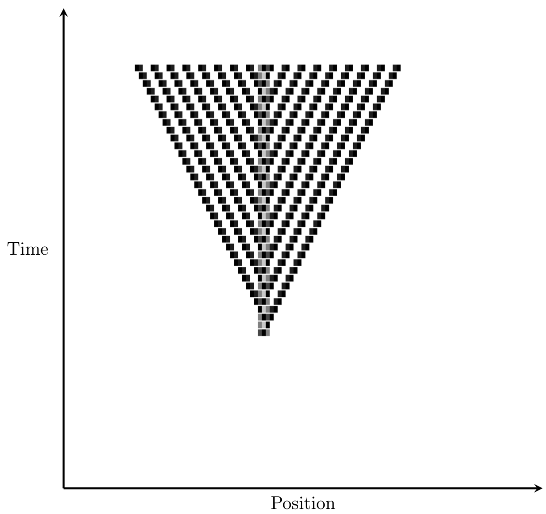

Figure 8: 1D Spiral

The one-dimensional spiral is a variant of the one-dimensional target pattern. The defect line is still vertical, however, new pulses are not created in pairs. A pulse is only generated in one direction each period, with the direction switching each time. It can also be observed in some solutions to the Belousov-Zhabotinskii reaction[5].

2.4.4 The Construction

Let $t \in \mathbb{N}_{>0}$ be the number of cells between the excited cells that alternate between spawning a diagonal line to either the left or the right. Then the automaton exists:
Let $S_t : \{0, \cdots, 4+t\}^{\mathbb{Z}} \to \{0, \cdots, 4+t\}^{\mathbb{Z}}$

$$(a_n)_{n\in\mathbb{Z}} \mapsto \left(\begin{cases} 1, & a_n = 4+t \vee (a_{n-1}, a_n, a_{n+1}) \in c(\{(1,0,0), (1,4,0), (2,0,1)\}) \\ 2, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(3,1,0), (2,1,0)\}) \\ 3, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(5,1,0)\} \uplus \{6, \cdots, \infty\} \times \{3\} \times \mathbb{N}_0) \\ 4, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(1,3,0), (1,3,1), (1,3,2)\} \uplus \{5, \cdots, \infty\} \times \{4\} \times \mathbb{N}_0) \\ 5, & (a_{n-1}, a_n, a_{n+1}) \in c(\{(4,1,3)\}) \\ a_n + 1, & 5 \leqslant a_n < 4+t \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$$

This exhibits the desired behaviour given the initial state:

$$\left(\begin{cases} 3, & n = -1 \\ 1, & n = 0 \\ 4, & n = 1 \\ 0, & \text{otherwise} \end{cases} \right)_{n\in\mathbb{Z}}$$

The above rule alternates between generating diagonal lines from the left or from the right with a period of t cells in between. The central idea is that the column of the points at positions $-1, 0, 1$ is $2t+2$-periodic, while outside of it, the states evolve like in Greenberg-Hastings. The periodic behaviour of the column is as follows: $(3,1,4) \to (4,5,1) \to (4,6,3) \to \cdots \to (4, 4+t, 3) \to (4,1,3)$. Then, the reflected version happens. It should be noted that in the case $t = 1$, that the $4,6,3) \to \cdots \to (4, 4+t, 3)$ part is skipped.

The 3 and 4 states together encode direction/asymmetry by swapping their positions every $t+1$ cells, with 4 generating excited states on the positions $-1, 1$ when a 1 is neighboring it, and 3 blocking unwanted excited states from being generated. The states $5, \cdots, 4+t$ are responsible for the length of the column.

2.4.5 Annihilation

The rule, assuming the central columns are far enough apart, does not allow for multiple columns to annihilate.

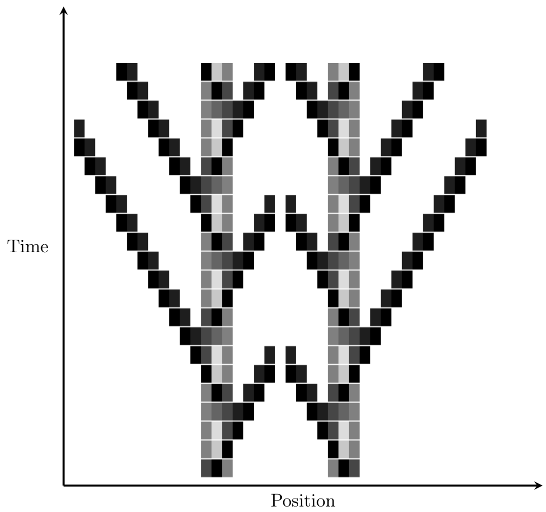

Figure 9: Two inward-facing initial states, with 9 cells inbetween

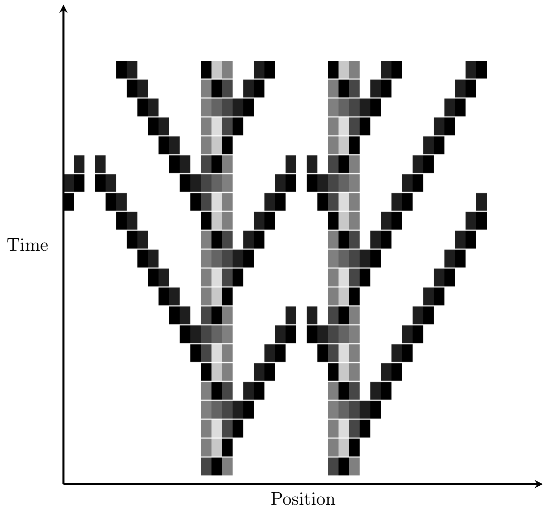

Figure 10: Two right-facing initial states, with 9 cells inbetween

2.4.6 Correctness

The grid $(c_n^m)_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0}$, defined by

$$c_n^m := \begin{cases} 1, & (m-|n|) \equiv 0(\mod 2t+2) \wedge m \geqslant n \geqslant 0 \text{ (right-moving wavetrains as for GH)} \\ 1, & (m-|n|) \equiv t+1(\mod 2t+2) \wedge t+1-m \leqslant n \leqslant 0 \text{ (left-moving wavetrains as for GH)} \\ 0, & \text{otherwise} \end{cases}$$

encodes which cells at position n after m iterations of S_t (with the starting conditions described in the construction) are excited (a value of 1) or not (a value of 0). This notably collapses the $3+t$ helper states into the rest state, and is distinct from the grid generated

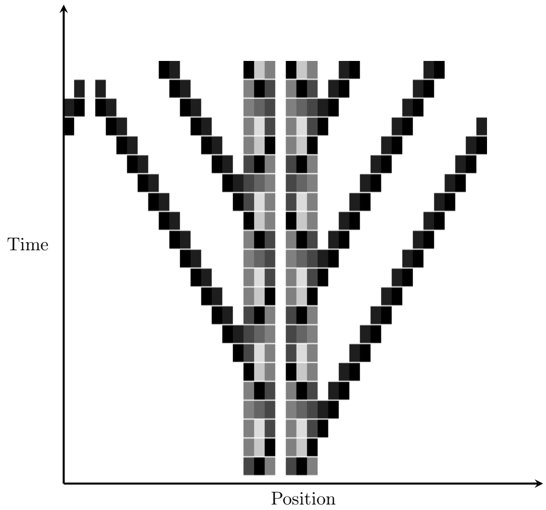

Figure 11: Two right-facing initial states, with 1 cell inbetween

by iterating S_t. It is, however, useful for understanding (and proving) the behaviour of the excited states. For a proof of the $c_{n,m}$'s correctness, see the appendix.

It can be easily seen from $c_{0,m}$ that the middle cell (that is, the cell at position 0) is excited exactly every t+1 cells, and thus has a gap of t non-excited cells inbetween, as required.
Additionally, for $n \in \mathbb{N}_{>0}, m \in \mathbb{N} \quad c_{n,m} = 1 \rightarrow c_{n+1,m+1} = 1$ and $c_{-n,m} = 1 \rightarrow c_{-(n+1),m+1} = 1$, which is the desired behaviour of the wavetrains. Lastly, if the middle cell is excited, in the next time step, either the left or the right neighbouring cell will be excited, switching sides from the last time.

Topological Entropy

The above automata, since they have more states than the three of GH, and many more transition rules, are more complex. This statement can be be made more rigorous by saying that their topological entropy is bounded from below by the topological entropy of GH:
Let $Z := \{(a_n)_{n\in\mathbb{Z}} | \quad \exists p \in \mathbb{Z} \cup \{-\infty, \infty\} : \forall k \in \mathbb{Z} : (k \geqslant p \rightarrow (a_k = 0 \rightarrow a_{k+1} \in \{0,1\}) \wedge (a_k = 1 \rightarrow a_{k+1} = 2) \wedge (a_k = 2 \rightarrow a_{k+1} = 0)) \wedge (k \leqslant p \rightarrow (a_k = 0 \Rightarrow a_{k-1} \in \{0,1\}) \wedge (a_k = 1 \rightarrow a_{k-1} = 2) \wedge (a_k = 2 \rightarrow a_{k-1} = 0))\}$
This set contains exactly those states which either consist only of the right-moving waves $0, 1, 2$, only the left-moving waves $2, 1, 0$, or both, in which case there exists a cell (at position p in the above definition) such that all cells to the left include only right-moving waves and all to the right only left-moving ones. That is, waves that move in opposite directions must move towards each other, and there is a clear middle that separates them. That this set is invariant under Greenberg-Hastings, and that the restriction of any of

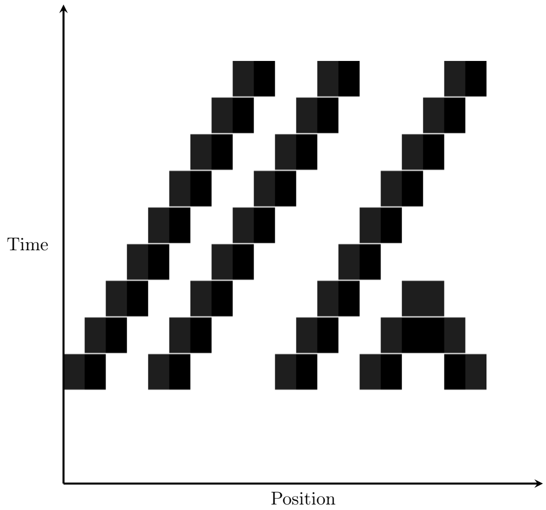

Figure 12: Example of configurations in Z

the rules below to Z is equal to the restriction of the Greenberg-Hastings rule to Z, can be easily checked.
It turns out[4], that all of the entropy of GH lies on the set Z. The other automata have the same behaviour on Z, but can also have dynamics outside of it, and thus their topological entropy is bounded from below by the topological entropy of GH.

References

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Appendix

Introducing Notation

In the following proofs, an expression of the form $A \Rightarrow \begin{cases} B \\ C \\ D \end{cases}$ is meant to state that B, C, D follow from A. If the implication arrow is inside the braces, the same is meant. As the following proofs will often require the collection of many intermediate statements, a way to capture this non-linear or branching behaviour was required.

The convention of writing $A \overset{C}{\Rightarrow} B$ in order to compactly write that B follows from A for reason C, will, in the following, be generalised to relations on non-logical objects, such that, for example, if $a \in A$ for reason φ, then it will be written as $a \overset{\varphi}{\in} A$. As the proofs will be of a rather long and technical nature, it is of great value to have compact notation to reduce the page space they take up.
The shorthand Def will be cited as a reason whenever the statement follows immediately from the definition of the explicit solution.

The proofs below are both proofs by induction over the time, m to prove that the explicit solution $((a_n^m)_{n\in\mathbb{Z}})$ given for time m is equal to the m-th iteration of the automaton $((b_n^m)_{n\in\mathbb{Z}})$. Due to the many conditionals in the rules of the automata, the proofs are split into many cases. Within each of these cases, several statements must be proven (that is, $a_n^{m+1} = b_n^{m+1}$ for each $n \in \mathbb{Z}$), and each of these statements requires knowledge of the values of $a_{n-1}^m, a_n^m, a_{n+1}^m$, and thus a large portion of the proof is preoccupied with the collection of various intermediate statements in order to identify values at time m to identify values at time $m+1$. When a part of the equality $a_n^{m+1} = b_n^{m+1}$ has been proven, it is boxed. The following proofs will not explicitly acknowledge that a case is finished because all relevant statements have been proven. Instead, the proof moves to the next case and the reader can readily check the boxed statements within the case to confirm that everything has been proven. Thus, the induction step of the proof can be compared to a computer program which utilises a divide-and-conquer approach to collect data (the intermediate statements) as it divides the problem into more specialised cases, until the data can be processed (the desired result has been derived from them).

Proof for pulse-replication

In the following, n is the position of a cell, m is time (how many times the automaton has been iterated) and k is how often the pattern has completely self-replicated already.
$\mathrm{wave}(n, m, k) :\equiv (m - k \cdot h = n - 2 - k \cdot w \geqslant 0)$
This codes the waves of excited states.
$\mathrm{vert}(n, m) :\equiv (\exists k \in \mathbb{N}_0 : (2 + k \cdot w = n \wedge 0 < m - k \cdot h \leqslant h - w))$
This encodes the helper states climbing vertically.
$\mathrm{diag}(n, m) :\equiv (\exists k \in \mathbb{N}_0 : h - w < m - k \cdot h \leqslant h - 1 \wedge ((m - k \cdot h) - (n - k \cdot w) = h - w - 1)$
This encodes the helper states climbing diagonally.

$$(a_n^m)_{n \in \mathbb{N}_0} := \begin{cases} 1, & \exists k \in \mathbb{N}_0 : \mathrm{wave}(n, m, k) \\ 2, & \exists k \in \mathbb{N}_0 : \mathrm{wave}(n + 1, m, k) \vee (\mathrm{wave}(n + 2, m, \frac{m}{h}) \wedge 0 = m \mod h) \\ 3, & \mathrm{vert}(n + 1, m) \vee \mathrm{diag}(n + 1, m) \\ (m \mod h) + 2, & (\mathrm{vert}(n, m) \vee \mathrm{diag}(n, m)) \wedge 1 \neq m \mod h \\ 0, & \text{otherwise} \end{cases}$$

$$(b_n^m)_{n \in \mathbb{N}} := F_{h,w}^m(\left(\begin{cases} 2, & n = 0 \\ 2, & n = 1 \\ 1, & n = 2 \\ 0, & \text{otherwise} \end{cases} \right)_{n \in \mathbb{Z}})$$

It can be easily seen that the above is a well-defined function. One can also easily check that the choice of m uniquely determines what n satisifies $\mathrm{vert}(n, m)$, if it exists. The same is true of diag.
Before proving the result of the automaton's correctness, proving a few Lemmas is necessary:
Lemma 1: $\forall n, m, k \in \mathbb{N}_0 : \mathrm{vert}(n + 1, m, k) \vee \mathrm{diag}(n + 1, m, k) \Rightarrow \forall K \in \mathbb{N}_0 \forall N \leqslant n + 1 : \neg \mathrm{wave}(N, m, K)$
(That is, there are no waves left to the trace).
Proof:
Let $n, m \in \mathbb{N}_0$.
That $\neg\mathrm{wave}(n + 1, m, k)$ follows immediately from $\mathrm{vert}(n + 1, m, k) \vee \mathrm{diag}(n + 1, m, k)$ (the equality from wave is contradicted by the strict inequality in vert or diag's condition that $h - w - 1 = 2$, when $h - w - 1 \geqslant 4$ by definition)
It remains to be shown for $N \leqslant n$:
For this, we first need to collect two statements, $(\star)$ and $(\star\star)$:
Case 1: $\mathrm{vert}(n, m)$:
$\Rightarrow \exists k \in \mathbb{N}_0 : 1 + k \cdot w = n \wedge 0 < m - k \cdot h \leqslant h - w$
$\Rightarrow k \cdot h < m$
$\Rightarrow k \cdot h - n < m - n$
$\overset{n=1+k \cdot w}{\Rightarrow} k \cdot (h - w) - 1 < m - n$
$\Rightarrow k \cdot (h - w) \leqslant m - n \quad (\star)$
$m - k \cdot h \leqslant h - w$
$\Rightarrow m - k \cdot h - h \leqslant -w$

$\overset{w>0}{\Rightarrow} m-(k+1)\cdot h<0 \quad (\star\star)$
Case 2: $\text{diag}(n+1,m)$:
$\Rightarrow \exists k \in \mathbb{N}_0 : h-w<m-k\cdot h \leqslant h-1 \wedge ((m-k\cdot h)-(n+1-k\cdot w)=h-w-1$
$\Rightarrow m-n+(-k\cdot h--k\cdot w)-1=h-w-1$
$\Rightarrow m-n=(k+1)\cdot(h-w)$
$\overset{h-w>0}{\Rightarrow} k\cdot(h-w)\leqslant m-n \quad (\star)$
$m-k\cdot h\leqslant h-1$
$\Rightarrow m-k\cdot h-h\leqslant -1$
$\Rightarrow m-(k+1)\cdot h<0 \quad (\star\star)$

Assume $\exists N\leqslant n, \exists K\in\mathbb{N}_0$ such that $\text{wave}(N,m,K)$:
$K\cdot(h-w)>K\cdot(h-w)-2 \overset{\text{wave}(N,m,K)}{=} m-N \overset{N\leqslant n}{\geqslant} m-n \overset{(\star)}{\geqslant} k\cdot(h-w)$
$\Rightarrow K>k\Rightarrow K\geqslant k+1$
$\text{wave}(m,N,K)\Rightarrow 0\leqslant m-K\cdot h \overset{K\geqslant k+1}{\leqslant} m-(k+1)\cdot h \overset{(\star\star)}{<} 0$ This is a contradiction, and thus $\text{wave}(N,m,K)$ is not satisfied.

Lemma 2: $\forall m,n,k\in\mathbb{N} : \text{wave}(n,m,k)\Rightarrow(\forall j\in\{1,\cdot,h-w-1\}\forall K\in\mathbb{N}_0 : \quad \neg\text{wave}(n+j,m,K))$ (That is, there is a gap of at least $h-w-1$ non-excited cells between any two excited cells)
Proof:
Let $n,m,k\in\mathbb{N}$ such that $\text{wave}(n,m,k)$:
$\Rightarrow m-k\cdot h=n-2-k\cdot w$
Assume $\exists K\in\mathbb{N}_0 : \exists j\in\{1,\cdots,h-w-1\} \quad \text{wave}(n+j,m,K)$:
$m-K\cdot h=n+j-2-K\cdot w$
$\Rightarrow n+j-2+K(h-w)=m=n-2+k(h-w)$
$\Rightarrow j=(k-K)(h-w)$
$\overset{0<j<h-w}{\Rightarrow} 0\leqslant(k-K)<1$
$\Rightarrow k-K=0$
$\Rightarrow j=0(h-w)=0\Rightarrow$ Contradiction to $j\in\{1,\cdots,h-w-1\}$

Lemma 3: Let $m\in\mathbb{N}_0, K:=\frac{m}{h}\in\mathbb{N}_0$. Then $\forall n,k\in\mathbb{N}_0 : (\text{wave}(n,m,k)\rightarrow n\geqslant K\cdot w+2\wedge k\leqslant K)$ (this Lemma is used when the pattern starts replicating again to establish (roughly speaking) where (the inequality on n) and how many waves there are $(K+1=|\{0,\cdots,K\}|)$)
Proof:
$\overset{\text{Def}}{\Rightarrow} 0\leqslant m-k\cdot h=n-2-k\cdot w$
Assume $n<K\cdot w+2$:
$\Rightarrow 0\leqslant m-k\cdot h=n-2-k\cdot w\leqslant K\cdot w+2-2-k\cdot w$
$\Rightarrow 0\leqslant m-k\cdot h\leqslant(K-k)\cdot w$
$\Rightarrow 0\leqslant(K-k)w$
$\Rightarrow 0\leqslant K-k\Rightarrow k\leqslant K$
$\Rightarrow n=m-k\cdot(h-w)+2 \overset{k\leqslant K,h-w>0}{\geqslant} m-K\cdot(h-w)+2=m-m+K\cdot w+2=K\cdot w+2$

Thus $n \geqslant K \cdot w + 2$:
$(K-k)h = Kh - kh \overset{K=\frac{m}{h}}{=} m - kh \overset{\text{wave}(n,m,k)}{=} n-2-k\cdot w \geqslant K\cdot w+2-2-k\cdot w = (K-k)w$
$\overset{h>w}{\Rightarrow} K-k \geqslant 0 \Rightarrow k \leqslant K$

Lemma 4: Let $m, N \in \mathbb{N}_0 : (a_n^m)_{n\in\mathbb{N}_0} = (b_n^m)_{n\in\mathbb{N}_0} \wedge N > w \cdot \lfloor \frac{m}{h} \rfloor + 2 \wedge \forall n \geqslant N :$ $a_n^m \in \{0,1,2\}$.
Then $\forall n \geqslant N : a_{n+1}^{m+1} = a_n^m = b_{n+1}^{m+1}$ (This lemma will be used in the induction step to prove equality to the right of the (right-moving) defect line)
Proof:
Let $n \geqslant N$.
Let $k_0 \in \mathbb{N}_0$ such that wave(n,m,k_0) :
$\Rightarrow m - k_0 \cdot h \geqslant 0 \Rightarrow k_0 \leqslant \lfloor \frac{m}{h} \rfloor$
$\Rightarrow n - 2 - k_0 \cdot w \overset{k_0 \leqslant \lfloor \frac{m}{h} \rfloor}{\geqslant} n - 2 - w \cdot \lfloor \frac{m}{h} \rfloor \overset{n\geqslant N}{\geqslant} N - 2 - w \cdot \lfloor \frac{m}{h} \rfloor \overset{\text{Assumption}}{>} 0$
$\Rightarrow ((\exists k \in \mathbb{N}_0 : m - k\cdot h = n - 2 - k \cdot w) \Leftrightarrow \exists k \in \mathbb{N}_0 : \text{wave}(n,m,k)) \quad (\star)$, that is, the inequality condition of wave can be dropped in this case.
Let $j \in \{0,1\}$:
$(\exists k \in \mathbb{N}_0 : \text{wave}(n+j,m,k))$
$\overset{(\star)}{\Leftrightarrow} (\exists k \in \mathbb{N}_0 : m - k\cdot h = (n+j) - 2 - k\cdot w)$
$\Leftrightarrow (\exists k \in \mathbb{N}_0 : m + 1 - k\cdot h = (n+j) + 1 - 2 - k\cdot w)$
$\overset{(\star)}{\Leftrightarrow} (\exists k \in \mathbb{N}_0 : \text{wave}(n+j+1,m+1,k))$
That is, these cells have the same state as their right diagonal cell if either cell has state 1. $(\star\star)$
Assume $0 = m \mod h \wedge \text{wave}(n+2,m,\lfloor \frac{m}{h} \rfloor)$:
$\Rightarrow 0 \overset{\text{wave}(n+2,m,\lfloor \frac{m}{h} \rfloor)}{=} m - \frac{m}{h} \cdot h = n + 2 - 2 - \frac{m}{h} \cdot w \overset{n\geqslant N}{\geqslant} N - w \cdot \frac{m}{h} > 0$
$\Rightarrow$ Contradiction.
Thus, $a_n^m = 2 \Leftrightarrow (\exists k \in \mathbb{N}_0 : \text{wave}(n,m,k)) \quad (\star\star\star)$
That $(\star\star)$ is also true if 1 is substituted with 2 follows from $(\star\star\star)$. Since the only other admissible state is 0, $(\star\star)$, substituting 1 with 0, must also still hold.
$\Rightarrow a_{n+1}^{m+1} = a_n^m$
It remains to be shown: $a_n^m = b_{n+1}^{m+1}$:
It follows from $h - w - 1 \geqslant 2$ and Lemma 2 that at most one of $a_n^m, a_{n+1}^m, a_{n+2}^m$ can be 1.
It follows from $(\star\star\star)$ and setting $j = 1$ that the same is true of 2.
From the definition of the a_n^m and $(\star\star\star)$ it follows that every 1 has a 2 to its left, and every 2 a 1 to its right.
$\Rightarrow (b_n^m, b_{n+1}^m, b_{n+2}^m) = (a_n^m, a_{n+1}^m, a_{n+2}) \in \{(0,0,0),(1,0,0),(2,1,0),(0,2,1),(0,0,2)\}$
In each of these cases, it can be easily seen that b_{n+1}^{m+1} takes on the value to the left, a_n^m.

Theorem 1: $a_n^m = b_n^m \forall n, m \in \mathbb{N}_0$
Proof by Induction over m: $a_n^m = b_n^m$:
$m = 0$:

$\exists k = 0 : 0 - 0 \cdot h = 2 - 2 - 0 \cdot w \geqslant 0 \Rightarrow \text{wave}(2, 0)$

$$\Rightarrow \begin{cases} \text{wave}(2,0) \Rightarrow \boxed{a_2^0 = 1 = b_2^0} \\ \text{wave}(1+1,0) \Rightarrow \boxed{a_1^0 = 2 = b_1^0} \\ \text{wave}(0+2,0) \overset{0=0 \mod h}{\Rightarrow} \boxed{a_0^0 = 2 = b_0^0} \end{cases} \qquad m = 0 \Rightarrow 0 \nless m \mod h \Rightarrow a_n^0 \neq 3 \forall n \in \mathbb{N}$$

$\forall N > 2 :$

$h > w \Rightarrow \neg\text{wave}(N, 0) \Rightarrow a_N^0 \notin \{1, 2\}$

$m = 0 \Rightarrow a_N^0 \neq (m \mod h) + 2 \Rightarrow a_N^0 = 0$

Induction Step:

$m \mapsto m + 1$:

In the following, the induction hypothesis will be used implicitly, that is, b_n^m and a_n^m will be used interchangeably.

$K := \lfloor \frac{m}{h} \rfloor$ (K encodes how often the pattern has already fully self-replicated by time m)

$r := m \mod h$ (r encodes at what vertical position in the pattern the state currently is)

Case 1: $0 = r(= m \mod h)$: (The initial state, or a shifted copy)

$h|m \overset{\text{Def}}{\Rightarrow} \forall n \in \mathbb{N}_0 : \neg(\text{vert}(n, m) \vee \text{diag}(n, m))$

$\overset{\text{Def}}{\Rightarrow} \text{wave}(K \cdot w + 2, m, K) \overset{\text{Def}}{\Rightarrow} a_{K \cdot w}^m = 2$

Lemma 3 $\Rightarrow \forall n < K{\cdot}w : (\nexists k \in \mathbb{N}_0 : \text{wave}(n{+}2, m, k) \vee \text{wave}(n{+}1, m, k) \vee \text{wave}(n, m, k)) \overset{\text{Def}a_n^m}{\Rightarrow}$
$a_n^m = 0$

$\Rightarrow (a_n^m, a_{n+1}^m, a_{n+2}^m) \in \{(0, 0, 0,), (0, 0, 2), (0, 2, 1)\} \Rightarrow b_{n+1}^{m+1} = 0 = a_{n+1}^{m+1}$

$\Rightarrow \boxed{a_n^{m+1} = b_n^{m+1} \quad (n \leqslant K \cdot w)}$

$(a_{K \cdot w}^m, a_{K \cdot w+1}^m, a_{K \cdot w+2}^m) = (2, 2, 1) \Rightarrow \boxed{b_{K \cdot w+1}^{m+1} = 3 \overset{\text{vert}(K \cdot w+1+1, m+1)}{=} a_{K \cdot w+1}^{m+1}}$

Lemma 2 $\Rightarrow (a_{K \cdot w+1}^m, a_{K \cdot w+2}^m, a_{K \cdot w+3}^m) = (2, 1, 0) \Rightarrow \boxed{b_{K \cdot w+2} = 2 \overset{\text{wave}(K \cdot w+2+1, m+1, K)}{=} a_{K \cdot w+2}^{m+1}}$

Lemma 2 $\Rightarrow (a_{K \cdot w+2}^m, a_{K \cdot w+3}^m, a_{K \cdot w+4}^m) = (1, 0, 0) \Rightarrow \boxed{b_{K \cdot w+3} = 1 \overset{\text{Def wave}}{=} a_{K \cdot w+3}^{m+1}}$

Lemma 4, with $N := K \cdot w + 3 \Rightarrow \boxed{a_n^{m+1} = b_n^{m+1} \quad (n \geqslant K \cdot w + 4)}$

Case 2: $0 < r \leqslant h - w$: (In this case, the helper states are part of a purely vertical (unmoving spatially) line)

$\Rightarrow \text{vert}(2 + K \cdot w, m)$

$$\Rightarrow \begin{cases} a_{1+K \cdot w}^m = 3 \\ a_{2+K \cdot w}^m = r + 2 \end{cases}$$

Lemma 1 $\Rightarrow \forall n \leqslant 1 + K \cdot w : a_n^m = 0$

$\Rightarrow \forall n \leqslant K \cdot w : (a_{n-1}^m, a_n^m, a_{n+1}^m) \in \{(0, 0, 0), (0, 0, 3)\} \Rightarrow b_n^{m+1} = 0$

It can be easily seen that $m + 2 - K(h - w)$ is the least value of a n_0 such that $\exists k \in \mathbb{N}_0 : \text{wave}(n_0, m, k)$

Thus, since vert has already been satisfied: $\forall n \in \{K \cdot w + 3, \cdots, m - K(h - w)\} = \{K \cdot w + 3, \cdots, K \cdot w + r\} : a_n^m = 0(\star)$

Case 2.1: $r < h - w$: (In the next timestep, the defect line continues to move only vertically/ is stationary in space)

$\Rightarrow \mathrm{vert}(2+K\cdot w, m+1)$
Case 2, substituting m with $m+1 \Rightarrow \boxed{a_n^{m+1} = 0 = b_n^{m+1} \quad (n \leqslant K\cdot w)}$
$(a_{K\cdot w}^m, a_{K\cdot w+1}^m, a_{K\cdot w+2}^m) = (0,3,r+2) \Rightarrow \boxed{b_{1+K\cdot w}^{m+1} = 3 = a_{1+K\cdot w}^{m+1}}$
$a_{K\cdot w+2}^m, = r+2 \Rightarrow \boxed{b_{2+K\cdot w}^{m+1} = r+1+2 \overset{\text{Case 2}}{=} a_{2+K\cdot w}^{m+1}}$
$\forall n \in \{K\cdot w+3, \cdots, m-K(h-w)\} : (a_{n-1}^m, a_n^m, a_{n+1}) \in \{(r+2,0,0),(r+2,0,2),(r+2,2,1),(0,0,0),(0,0,2),(0,2,1)\}$
$\Rightarrow \boxed{b_n^{m+1} = 0 \overset{(\star)\text{ substituting } m \text{ with } m+1}{=} a_n^{m+1} \quad (n \in \{K\cdot w+3, \cdots, m-K(h-w)\})}$
Case 2.2: $r = h-w$: (The system transitions between the purely vertical component of the defect line into the diagonal component)
$r = h-w \geqslant 3 \Rightarrow K\cdot w+3 \in \{K\cdot w+3, \cdots, K\cdot w+r\} \overset{(\star)}{\Rightarrow} a_{K\cdot w+3}^m = 0$
$\Rightarrow (a_{K\cdot w+1}^m, a_{K\cdot w+2}^m, a_{K\cdot w+3}^m) = (3, h-w+2, 0)$
Case 2.2.1: $w = 1$ (One cell of spatial movement is already taken care of without needing the diagonal line, so in this case, the diagonal line is skipped, and must be treated separately)
Works analogously to Case 3.2 (which covers the transition from the diagonal to a shifted copy of the intial state).
Case 2.2.2: $w > 1$:
$\Rightarrow \boxed{b_{K\cdot w+2}^{m+1} = 3 \overset{\mathrm{diag}(K\cdot w+2+1, m+1)}{=} a_{K\cdot w+2}^{m+1}} \Rightarrow (a_{K\cdot w}^m, a_{K\cdot w+1}^m, a_{K\cdot w+2}^m) = (0,3,h-w+2) \Rightarrow$
$\boxed{b_{K\cdot w+1}^{m+1} = 0 \overset{a_{K\cdot w+2}^{m+1}=3 \wedge \text{Lemma 1}}{=} a_{K\cdot w+1}^{m+1}}$
$(a_{K\cdot w+2}^m, a_{K\cdot w+3}^m, a_{K\cdot w+4}^m) \in \{(h-w+2,0,0),(h-w+2,0,2)\} \Rightarrow \boxed{b_{K\cdot w+3}^{m+1} = h-w+3 \overset{\mathrm{diag}(K\cdot w+3, m+1)}{=} a_{K\cdot w+3}^{m+1}}$
Lemma 4, with $N := K\cdot w+3 \Rightarrow \boxed{b_n^{m+1} = a_n^{m+1} \quad (n \geqslant K\cdot w+4)}$

Case 3: $r > h-w$: (The case where the system is in the diagonal component of the defect line)
$(m-K\cdot h)-(m+1-(K+1)(h-w)-K\cdot w) = (h-w)-1 \Rightarrow \mathrm{diag}(m+1-(K+1)(h-w), m)$
$$\Rightarrow \begin{cases} a_{m-(K+1)(h-w)} = 3 \\ a_{m+1-(K+1)(h-w)} = r+2 \end{cases} \quad (\star\star\star)$$
Lemma 1$\Rightarrow \forall n < m-(K+1)(h-w) : a_n^m = 0(\star)$
$\Rightarrow (a_{n-1}^m, a_n^m, a_{n+1}^m) \in \{(0,0,0),(0,0,3)\} \Rightarrow b_n^{m+1} = 0(\star\star)$
Assume $\exists k \in \mathbb{N}_0, j \in \{0,1\} : \mathrm{wave}(m+j+2-(K+1)(h-w), m, k)$:
$\Rightarrow m-k\cdot h = m+j+2-(K+1)(h-w)-2-k\cdot w \geqslant 0$
$\Rightarrow 0 = j-(K+1)(h-w)+k\cdot(h-w)$
$\Rightarrow (K+1-k)(h-w) = j \in \{0,1\}$
$\overset{h-w>1 \wedge (K+1),(h-w)\in\mathbb{N}_0}{\Rightarrow} (K+1-k)(h-w) = 0 = j$
$\Rightarrow K+1-k = 0 \Rightarrow k = K+1$
$\Rightarrow m-k\cdot h \geqslant 0$
$\overset{K=\lfloor \frac{m}{h} \rfloor}{\Rightarrow} K+1 = k \leqslant K \Rightarrow$ Contradiction.

$\Rightarrow a^m_{m+2-(K+1)(h-w)} = 0$

Case 3.1: $r < h-1$: (In this case, the next step is still part of the diagonal defect line) $\Rightarrow$ Case 3, substituting m with $m+1$, holds also. $(\star) \Rightarrow \forall n < m+1-(K+1)(h-w)$: $a^{m+1}_n = 0$

$\Rightarrow \boxed{a^{m+1}_n = 0 \overset{(\star\star)}{=} b^{m+1}_n \quad (n < m-(K+1)(h-w))}$

$\mathrm{diag}(m-(K+1)(h-w), m) \Rightarrow a^m_{m-(K+1)(h-w)} = 3, a^m_{m-(K+1)(h-w)+1} = r+2 \ \forall k \in \mathbb{N}_0, \forall j \in \{0,1\}$:

$m - kh = (m-(K+1)(h-w)+j+2) - 2 - kh$

$\Rightarrow k(h-w)+j = (K+1)(h-w) \Rightarrow j = 0 \Rightarrow k = K+1 \Rightarrow m-kh = m-(K+1)h < 0 \Rightarrow \neg\mathrm{wave}(m-(K+1)(h-w)+2, m, k) \Rightarrow a^m_{m-(K+1)(h-w)+2} = 0, a^m_{m-(K+1)(h-w)+3} \in \{0,2\}$

$$\Rightarrow \begin{cases} (a^m_{m-(K+1)(h-w)}, a^m_{m-(K+1)(h-w)+1}, a^m_{m-(K+1)(h-w)+2}) = (3, r+2, 0) \Rightarrow b^{m+1}_{m-(K+1)(h-w)+1} = 3 \\ [(a^m_{m-(K+1)(h-w)+1}, a^m_{m-(K+1)(h-w)+2}, a^m_{m-(K+1)(h-w)+3}) \in \{(r+2,0,0),(r+2,0,2)\} \\ \Rightarrow b^{m+1}_{m-(K+1)(h-w)+2} = r+2+1] \end{cases}$$

$\mathrm{diag}(m-(K+1)(h-w)+2, m+1)$

$\Rightarrow a^{m+1}_{m-(K+1)(h-w)+1} = 3, a^{m+1}_{m-(K+1)(h-w)+2} = ((m+1) \mod h) + 2 = r+1+2$

$\Rightarrow \boxed{b^{m+1}_n = a^{m+1}_n \quad n \in m-(K+1)(h-w)+\{1,2\}}$

$m-(K+1)(h-w)+3 = r+Kh-(K+1)(h-w)+3 = r+3-(h-w)+Kw \overset{r>h-w}{>} Kw+2$

$\overset{\text{Lemma 4}}{\Rightarrow} \boxed{b^{m+1}_n = a^{m+1}_n \quad n \geqslant m-(K+1)(h-w)+3}$

$(\star) \Rightarrow (a^m_{m-(K+1)(h-w)-1}, a^m_{m-(K+1)(h-w)}, a^m_{m-(K+1)(h-w)+1}) = (0, 3, r+2)$

$\Rightarrow \boxed{b^{m+1}_{m-(K+1)(h-w)} = 0 = a^{m+1}_{m-(K+1)(h-w)}}$

Case 3.2: $r = h-1$: (In this case, the automaton transitions out of the diagonal component of the defect line into the shifted copy of the initial state)

$\Rightarrow K+1 = \frac{m+1}{h}$

$(\star\star\star) \Rightarrow$ diag has already been satisfied and since $m \mod h \neq 0$, for any $n \notin \{m-(K+1)(h-w), m-(K+1)(h-w)+1\}$: $(a^m_n \neq 0 \Leftrightarrow \exists j \in \{0,1\} : \exists k \in \mathbb{N}_0 : \mathrm{wave}(n+j, m, k))(\spadesuit)$

Lemma 1 $\Rightarrow \forall n \leqslant m-(K+1)(h-w) : \forall k \in \mathbb{N} : \neg\mathrm{wave}(n,m,k) \overset{\spadesuit}{\Rightarrow} \forall n < m-(K+1)(h-w)$: $a^m_n \Rightarrow (a^m_{n-1}, a^m_n, a^m_{n+1}) \in \{(0,0,0),(0,0,3)\} \Rightarrow b^{m+1}_n = 0 \overset{\text{Lemma 3, substituting } m \text{ with } m+1,\ K \text{ with } K+1}{=} a^{m+1}_n$

$\Rightarrow \boxed{b^{m+1}_n = 0 = a^{m+1}_n \quad n < m-(K+1)(h-w)}$

$(\star\star\star) \Rightarrow (a_{m-(K+1)(h-w)-1, m-(K+1)(h-w), m-(K+1)(h-w)+1}) = (0, 3, r+2)$

$\Rightarrow \boxed{b^{m+1}_{m-(K+1)(h-w)} = 2 \overset{\mathrm{wave}(m-(K+1)(h-w)+2, m+1, K+1) \wedge m+1 \mod h = 0}{=} a_{m-(K+1)(h-w)}}$

Suppose $\exists k \in \mathbb{N}_0, j \in \{0,1,2\} : \mathrm{wave}(m-(K+1)(h-w)+2+j, m, k)$:

$\Rightarrow m - kh = m-(K+1)(h-w)+2+j-2-kw$

$\Rightarrow (K+1)(h-w) = j + k(h-w)$

$\overset{h-w>2}{\Rightarrow} j = 0 \Rightarrow (K+1)(h-w) = k(h-w)$

$\Rightarrow K+1 = k \Rightarrow m-kh = m-(K+1)h = m-(m+1) < 0 \Rightarrow$ Contradiction to wave's requirement of m, k.

Thus, $(a^m_{m-(K+1)(h-w)}, a^m_{m-(K+1)(h-w)+1}, a^m_{m-(K+1)(h-w)+2}, a^m_{m-(K+1)(h-w)+3}) = (3, r+2, 0) \Rightarrow \boxed{b^{m+1}_{m-(K+1)(h-w)+1} = 2 \overset{\text{wave}(m-(K+1)(h-w)+1+1,m+1,K+1)}{=} a^{m+1}_{m-(K+1)(h-w)+1}}$

$\boxed{b^{m+1}_{m-(K+1)(h-w)+2} = 1 \overset{\text{wave}(m-(K+1)(h-w)+2,m+1,K+1)}{=} a^{m+1}_{m-(K+1)(h-w)+2}}$

$m-(K+1)(h-w)+3 = m - \frac{m+1}{h}(h-w)+3 = \frac{m+1}{h}w+2 = (\lfloor\frac{m}{h}\rfloor+1)w+2 > \lfloor\frac{m}{h}\rfloor w+2 \overset{\text{Lemma 4}}{\Rightarrow} \boxed{a^{m+1}_n = b^{m+1}_n \quad (n \geqslant m-(K+1)(h-w)+3)}$

1D Spiral

In order to talk about the periodic behaviour and other required properties of this automaton, the space-time grid $(b^m_{n\in\mathbb{Z}})_{m\in\mathbb{N}_0}$ defined by $(b^m_n)_{n\in\mathbb{N}} := S^m_t(\left(\begin{cases} 3, & n=-1 \\ 1, & n=0 \\ 4, & n=1 \\ 0, & \text{otherwise}\end{cases}\right)_{n\in\mathbb{Z}})$ has been transformed into an explicit form $((a^m_n)_{n\in\mathbb{Z}})$, with the state of grid cells being dependent only on n, m:

$$\forall m\in\mathbb{N}_0: \quad (a^m_n)_{n\in\mathbb{Z}} := \left(\begin{cases} \begin{cases} 3, & \exists k\in\{t+3,\cdots,2t+2\}: \quad m\equiv k(\mod 2t+2) \\ 4, & \exists k\in\{1,\cdots,t+1\}: \quad m\equiv k(\mod 2t+2) \\ 1, & m\equiv(t+2)(\mod 2t+2)\end{cases}, & n=-1 \\ \begin{cases} 1, & m\equiv 0(\mod t+1) \\ (m \mod (t+1))+4, & \text{otherwise}\end{cases}, & n=0 \\ \begin{cases} 3, & \exists k\in\{2,\cdots,t+1\}: \quad m\equiv k(\mod 2t+2) \\ 4, & \exists k\in\{t+2,\cdots,2t+2\}: \quad m\equiv k(\mod 2t+2) \\ 1, & m\equiv 1(\mod 2t+2)\end{cases}, & n=1 \\ \begin{cases} 1, & (m-n)\equiv 0(\mod 2t+2) \\ 2, & (m-n)\equiv 1(\mod 2t+2) \\ 0, & \text{otherwise}\end{cases}, & m\geqslant n\geqslant 2 \\ \begin{cases} 1, & (m+n-t-1)\equiv 0(\mod 2t+2) \\ 2, & (m+n-t-1)\equiv 1(\mod 2t+2) \\ 0, & \text{otherwise}\end{cases}, & t+1-m\leqslant n\leqslant -2 \\ 0, & \text{otherwise}\end{cases}\right)_{n\in\mathbb{Z}}$$

This explicit form, however, because it needs to capture the entire machinery of the automaton, is rather opaque.

The grid $(c^m_n)_{(n,m)\in\mathbb{Z}\times\mathbb{N}_0}$, defined by

$$c^m_n := \begin{cases} 1, & (m-|n|)\equiv 0(\mod 2t+2)\wedge m\geqslant n\geqslant 0 \text{ (right-moving wavetrains as for GH)} \\ 1, & (m-|n|)\equiv t+1(\mod 2t+2)\wedge t+1-m\leqslant n\leqslant 0 \text{ (left-moving wavetrains as for GH)} \\ 0, & \text{otherwise}\end{cases}$$

is the image under the mapping that maps all non-excited states to 0. From this, the periodic behavior can be much more easily seen.

Proof of validity of explicit solution

The statement $\forall m \in \mathbb{N}_0 (a_n^m)_{n\in\mathbb{Z}} = (b_n^m)_{n\in\mathbb{Z}}$ will now be proven.
In the following, a state where the central column is either the same as that of the initial state or its reflected version, (that is, a state which locally looks like the start) will be referred to as a *beginning* state.
The proof of the statement will be a proof by induction over the time, m. Since the values of the explicit grid (a_n^m) depend heavily on what values $m \mod (t+1)$ and $m \mod (2t+2)$ have, the proof must be split into cases during the induction step depending on whether these moduli are 0 or not. This splitting into cases corresponds with whether the pattern has been finished and must start again, self-replicating a reflected version $(0 \equiv m(\mod (t+1)))$, or is still unfinished $(0 \not\equiv m \mod (t+1))$.
Before proving the complete result, two lemmas need to be proven:

Lemma 1. $\forall m \in \mathbb{N} : (a_n^m)_{n\in\mathbb{Z}} = (b_n^m)_{n\in\mathbb{Z}} \to (a_{-1}^{m+1}, a_0^{m+1}, a_1^{m+1}) = (b_{-1}^{m+1}, b_0^{m+1}, b_1^{m+1})$, *that is, the equalities hold in the middle column in the next time step*

Proof. In the following, whenever one of the three equalities which are part of this Lemma's statement are proven, they are boxed. Thus, the progress of any given case can be tracked by how many boxes have already appeared in it, with three boxes marking the end of a finished case.

Case 1: $0 = m \mod (t+1)$: (This case covers the *beginning* states)

Case 1.1: $0 = m \mod 2t+2$ (the *beginning* of a non-reflected version of the initial state):

$$\begin{cases} \overset{\text{Def}}{\Rightarrow} a_{-1}^m = 3 \wedge a_1^m = 4 \wedge a_1^{m+1} = 1 \wedge a_{-1}^{m+1} = 4 \\ \Rightarrow m-2 \mod (2t+2) = -2 \mod (2t+2) = 2t \overset{t\geqslant 1}{\notin} \{0,1\} \overset{\text{Def}}{\Rightarrow} a_2^m = 0 \\ 1 = (m+1) \mod (2t+2) \overset{\text{Def}}{\Rightarrow} a_1^{m+1} = 1 \end{cases}$$

$$\Rightarrow (a_0^m, a_1^m, a_2^m) = (1,4,0) \Rightarrow \boxed{b_1^{m+1} = 1 = a_1^{m+1}}$$

$$a_{-2}^m \in \{0,1,2\} \Rightarrow (a_{-2}^m, a_{-1}^m, a_0^m) \in \{(0,3,1),(1,3,1),(2,3,1)\} \Rightarrow \boxed{b_{-1}^{m+1} = 4 = a_{-1}^{m+1}}$$

Case 1.2: $m \equiv (t+1) \mod (2t+2)$ (the reflected version):

$$\begin{cases} \overset{\text{Def}}{\Rightarrow} a_{-1}^m = 4 \wedge a_1^m = 3 \wedge a_{-1}^{m+1} = 1 \wedge a_1^{m+1} = 4 \\ \Rightarrow (m + -2 - t - 1) \mod (2t+2) = (t+1-t-1-2) \mod (2t+2) = 2t \overset{t\geqslant 1}{\notin} \{0,1\} \overset{\text{Def}}{\Rightarrow} a_{-2}^m = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (a_{-2}^m, a_{-1}^m, a_0^m) = (0,4,1) \Rightarrow \boxed{b_{-1}^{m+1} = 1 = a_{-1}^{m+1}} \\ (a_0^m, a_{-1}^m, a_2^m) \in \{(1,3,0),(1,3,1),(1,3,2)\} \Rightarrow \boxed{b_1^{m+1} = 4 = a_1^{m+1}} \end{cases}$$

(Case 1 continued):

$$0 \equiv m \mod (t+1) \begin{cases} \overset{\text{Def}}{\Rightarrow} a_0^m = 1 \\ \overset{t+1>1}{\Rightarrow} 1 = (m+1) \mod (t+1)) \overset{\text{Def}}{\Rightarrow} a_0^{m+1} = 5 \end{cases}$$

$$\Rightarrow (a_{-1}^m, a_0^m, a_1^m) \in c(\{(3,1,4)\}) \Rightarrow \boxed{b_0^{m+1} = 5 = a_0^{m+1}}$$

Case 2: $0 \neq k := m \mod (t+1)$: (k encodes how much time has passed since the last beginning state)

Case 2.1: $2t+1 = m \mod (2t+2)$: (The next state must be a beginning state, specifically which is not a reflected version of the initial state)

$$\begin{cases} \overset{\text{Def}}{\Rightarrow} a_{-1}^{m+1} = 3 \\ \Rightarrow m+1 \mod (t+1) = ((m+1) \mod (2t+2)) \mod (t+1) = 0 \overset{\text{Def}}{\Rightarrow} a_0^{m+1} = 1 \\ \overset{\text{Def}}{\Rightarrow} a_1^{m+1} = 4 \end{cases}$$

Case 2.1.1: $t = 1$: (In this case, the progression $(4,5,1) \to (4,6,3) \to \cdots \to (4,4+t,3)$ is skipped, as $4 + t = 5$ and thus must be dealt with separately)

$$\begin{cases} a_{-1}^m = 1 \\ \overset{\text{Def}}{\Rightarrow} a_0^m = 4+1 = 5 \wedge a_1^m = 4 \wedge a_0^{m+1} = 1 \\ \Rightarrow (m-2) \mod (2t+2) \overset{t=1}{=} 3-2 \mod 4 = 1 \overset{\text{Def}}{\Rightarrow} a_2^m = 2 \\ (m + -2 - t - 1) \mod (2t+2) \overset{t=1}{=} (3-2-1-1) \mod 4 = -1 \mod 4 = 3 \notin \{0,1\} \overset{\text{Def}}{\Rightarrow} a_{-2}^m = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (a_{-2}, a_{-1}^m, a_0^m) = (0,1,5) \Rightarrow \boxed{b_{-1}^{m+1} = 3 = a_{-1}^{m+1}} \\ (a_{-1}^m, a_0^m, a_1^m) = (1,5,4) \Rightarrow \boxed{b_0^{m+1} = 1 = a_0^{m+1}} \\ (a_0^m, a_1^m, a_2^m) = (5,4,2) \Rightarrow \boxed{b_1^{m+1} = 4 = a_1^{m+1}} \end{cases}$$

Case 2.1.2: $t > 1$:

$$\Rightarrow \begin{cases} 2t+1 \in \{t+3, \cdots, 2t+2\} \\ 2t+1 = m \mod (2t+2) \wedge k = m \mod (t+1) \Rightarrow k = t > 1 \Rightarrow a_0^m = k+4 \geqslant 6 \end{cases}$$

$$\overset{\text{Def}}{\Rightarrow} \begin{cases} a_{-1}^m = 3 \\ a_0^m = 4 + k \overset{t>1}{\geqslant} 6 \\ a_1^m = 4 \end{cases}$$

$$\Rightarrow \begin{cases} (a_{-2}^m, a_{-1}^m, a_0^m) \in c(\{(5,1,0)\} \uplus \{6,\cdots,\infty\} \times \{3\} \times \mathbb{N}_0) \Rightarrow \boxed{b_{-1}^{m+1} = 3 = a_{-1}^{m+1}} \\ a_0^m = 4+t \Rightarrow \boxed{b_0^{m+1} = 1 = a_0^{m+1}} \\ (a_{-2}^m, a_{-1}^m, a_0^m) \in c(\{(5,1,0)\} \uplus \{5,\cdots,\infty\} \times \{4\} \times \mathbb{N}_0) \Rightarrow \boxed{b_1^{m+1} = 4 = a_1^{m+1}} \end{cases}$$

Case 2.2: $2t+1 > k = m \mod (2t+2) > 0$: (In between the previous cases)

Case 2.2.1: $1 = m \mod (2t+2)$: (one time step after a beginning state that is not locally a reflected version of the initial state)

$$\begin{cases} \overset{\text{Def}}{\Rightarrow} a_{-1}^m = a_{-1}^{m+1} = 4 \wedge a_1^m = 1 \wedge a_1^{m+1} = 3 \\ \Rightarrow (m-2) \mod (2t+2) = 1-2 \mod (2t+2) = 2t+1 \overset{t \geqslant 1}{\notin} \{0,1\} \overset{\text{Def}}{\Rightarrow} a_2^m = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (a_{-2}^m, a_{-1}^m, a_0^m) \in c(\{(5,1,0)\} \uplus \{5,\cdots,\infty\} \times \{4\} \times \mathbb{N}_0) \Rightarrow \boxed{b_{-1}^{m+1} = 4 = a_{-1}^{m+1}} \\ (a_{-1}^m, a_0^m, a_1^m) = (4,5,1) \text{ (Relevant for the following two subcases)} \\ (a_0^m, a_1^m, a_2^m) = (5,1,0) \Rightarrow \boxed{b_1^{m+1} = 3 = a_1^{m+1}} \end{cases}$$

Case 2.2.1.1: $t = 1$:
$(a_{-1}^m, a_0^m, a_1^m) = (4,5,1) \Rightarrow \boxed{b_0^{m+1} = 1 = a_0^{m+1}}$

Case 2.2.1.2: $t > 1$:
$(a_{-1}^m, a_0^m, a_1^m) = (4,5,1) \Rightarrow \boxed{b_0^{m+1} = 6 = a_0^{m+1}}$

Case 2.2.2: $1 < k = m \mod (2t+2) \leqslant t+1$ (Now in between a non-reflected and a reflected beginning state)

$$\overset{\text{Case } 2/0 \neq m \mod (t+1)}{\Rightarrow} k = m \mod (2t+2) < t+1 \Rightarrow t+4 > t+1 \geqslant a_0^m \overset{\text{Def}}{\geqslant} 6$$

$$\overset{\text{Def}}{\Rightarrow} a_{-1}^m = a_{-1}^{m+1} = 4 \wedge a_1^m = a_1^{m+1} = 3$$

$$\Rightarrow \begin{cases} (a_{-2}^m, a_{-1}^m, a_0^m) \in c(\{(5,1,0)\} \uplus \{5,\cdots,\infty\} \times \{4\} \times \mathbb{N}_0) \Rightarrow \boxed{b_{-1}^{m+1} = 4 = a_{-1}^{m+1}} \\ t+4 > a_0^m \geqslant 5 \Rightarrow \boxed{b_0^{m+1} = a_0^m + 1 = a_0^{m+1}} \\ (a_0^m, a_1^m, a_2^m) \in c(\{(5,1,0)\} \uplus \{6,\cdots,\infty\} \times \{3\} \times \mathbb{N}_0) \Rightarrow \boxed{b_1^{m+1} = 3 = a_1^{m+1}} \end{cases}$$

Case 2.2.3: $t+2 = m \mod (2t+2)$:

$$\begin{cases} \Rightarrow m \mod (t+1) = 1 \overset{\text{Def}}{\Rightarrow} a_0^m = 4+1 = 5 \\ \overset{\text{Def}}{\Rightarrow} a_1^m = a_1^{m+1} = 4 \wedge a_{-1}^m = 1 \wedge a_{-1}^{m+1} = 3 \\ \Rightarrow (m-2-t-1) \mod (2t+2) = t+2-2-t-1 \mod (2t+2) = 2t+1 \notin \{0,1\} \Rightarrow a_{-2}^m = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (a_0^m, a_1^m, a_2^m) \in c(\{(5,1,0)\} \uplus \{5,\cdots,\infty\} \times \{4\} \times \mathbb{N}_0) \Rightarrow \boxed{b_{-1}^{m+1} = 4 = a_{-1}^{m+1}} \\ (a_{-1}^m, a_0^m, a_1^m) = (1,5,4) \text{ (again, required for the two following subcases below)} \\ (a_{-2}^m, a_{-1}^m, a_0^m) = (0,1,5) \Rightarrow \boxed{b_{-1}^{m+1} = 3 = a_{-1}^{m+1}} \end{cases}$$

$\boxed{b_0^{m+1} = a_0^{m+1}}$ follows analogously to the Cases 2.2.1.1 and 2.2.1.2.

Case 2.2.4: $t+2 < m \mod (2t+2):$
Works Analogously to 2.2.2. $\square$

Lemma 2. *$\forall m \in \mathbb{N}_0 : ((a_n^m)_{n\in\mathbb{Z}} = (b_n^m)_{n\in\mathbb{Z}} \Rightarrow b_{n+sgn(n)}^{m+1} = a_n^m \quad \forall n \in \mathbb{Z}, |n| \geqslant 2)$, that is, after one time step, outside the middle five cells, the state of a cell is equal to the state of the cell diagonally from it (one step in time, one in space). This basically states that the generated pulses/waves travel diagonally, just like for GH.*

Proof. Wlog assume $n \geqslant 2:$
Case 1: $a_n^m = 1:$

Then $\forall j \in \{1, \cdots, 2t-1\}: \quad a_{n+j}^m = 0.$ (that is, the next $2t-1$ cells to the right are at rest)
Suppose not:
Then $m-(n+j) \mod (2t+2) \in \{0,1\}$

Case $m-(n+j) \mod (2t+2) = 0:$
$\Rightarrow m-n-j \equiv m-n(\mod (2t+2)) \overset{2t+2>j>0}{\Rightarrow}$ contradiction

Case $m-(n+j) \mod (2t+2) = 1:$
$\Rightarrow 0 \equiv m-n \equiv m-n-j+j \equiv j+1 \mod (2t+2) \overset{2t+1>j>0}{\Rightarrow}$ contradiction

It can also be easily seen from the above second case that if $j = 2t$, that $a_{n+j}^m \in \{0,2\}$
Then $(a_n^m, a_{n+1}^m, a_{n+2}^m) \in c((1,0,0),(1,0,2)) \Rightarrow \boxed{b_{n+1}^{m+1} = 1 = a_n^m}$

Case $a_n^m = 2:$
Then $m-n \mod (2t+2) = 1$
$\Rightarrow m-(n+1) \mod (2t+2) = 0$
$\Rightarrow a_{n+1}^m = 1 \Rightarrow \boxed{b_{n+1}^{m+1} = 2 = a_n^m}$

Case $a_n^m = 0:$

Let $l \in \mathbb{N}$ such that $l \geqslant 2, a_{l+1}^m = 1$. Then $a_l^m = 2:$ (that is, a 2 must be to the left of a 1)
Proof: $0 \equiv m-(l+1)(\mod (2t+2)) \Rightarrow 1 \equiv m-l(\mod (2t+2)) \Rightarrow a_l^m = 2$

Assume $b_{n+1}^{m+1} = 2:$
$\Rightarrow a_{n+1}^m = 1 \Rightarrow a_n^m = 2 \Rightarrow$ contradiction
Assume $b_{n+1}^{m+1} = 1:$
$\Rightarrow (a_n^m, a_{n+1}^m, a_{n+2}^m) \in \{(1,0,0),(0,0,1),(1,0,2),(2,0,1)\}$
$\overset{a_n^m=0}{\Rightarrow} a_{n+2}^m = 1 \Rightarrow a_{n+1}^m = 2 \Rightarrow$ contradiction)

$\Rightarrow \boxed{b_{n+1}^{m+1} = 0 = a_n^m}$ $\square$

Theorem 1. *$(a_n^m)_{n\in\mathbb{Z}} = (b_n^m)_{n\in\mathbb{Z}} \quad \forall m \in \mathbb{N}$, that is, (a_n^m) is the explicit solution of (b_n^m).*

Proof. Induction over m:
$m = 0$: Follows directly from their definitions.

Induction Step: Assume $a_n^m = b_n^m \quad \forall n \in \mathbb{Z}$:

Case 1: $|n| \leqslant 1$: (middle column)
Lemma 1 $\Rightarrow \boxed{a_n^{m+1} = b_n^{m+1}}$

Case 2: $|n| > 2$: (outside, which should evolve as GH)
$a_n^{m+1} = a_{n-\mathrm{sgn}(n)+\mathrm{sgn}(n)}^{m+1} \overset{\mathrm{Def}}{=} a_{n-\mathrm{sgn}(n)}^{m} \overset{\text{Lemma 2}}{=} b_{n-\mathrm{sgn}(n)+\mathrm{sgn}(n-\mathrm{sgn}(n))}^{m+1} \overset{|n|>2}{=} b_{n-\mathrm{sgn}(n)+\mathrm{sgn}(n)}^{m+1} = b_n^{m+1}$
$\Rightarrow \boxed{a_n^{m+1} = b_n^{m+1}}$

Case 3: $n \in \{-2, 2\}$: (the two columns which have both the middle column and the outside as neighbours, and must thus be dealt with separately)
Wlog $n = 2$:
Case 3.1: $a_2^m = 0$:
$$\overset{\mathrm{Def}}{\Rightarrow} \begin{cases} 1 < m - 2 \mod (2t+2) < 2t+2 \\ a_2^{m+1} \in \{0,1\} \end{cases}$$
Case 3.1.1: $a_2^{m+1} = 0$
$\overset{\mathrm{Def}}{\Rightarrow} m-2 \mod (2t+2) < 2t+1 \Rightarrow m \not\equiv 1 \mod (2t+2) \Rightarrow a_1^m \neq 1 \Rightarrow \boxed{b_2^{m+1} = 0 = a_2^{m+1}}$
Case 3.1.2: $a_2^{m+1} = 1$:
$\overset{\mathrm{Def}}{\Rightarrow} 0 = m - 1 \mod (2t+2) \Rightarrow 1 = m \mod (2t+2) \Rightarrow (a_1, a_2, a_3) = (1, 0, 0) \Rightarrow \boxed{b_2^{m+1} = 1 = a_2^{m+1}}$
Case 3.2: $a_2^m = 1$:
$$\overset{\mathrm{Def}}{\Rightarrow} \begin{cases} 0 = (m-2) \mod (2t+2) \Rightarrow 1 = (m-1) \mod (2t+2) \Rightarrow a_2^{m+1} = 2 \\ a_1^m = 3 \Rightarrow (a_1^m, a_2^m, a_3^m) = (3, 1, 0) \Rightarrow b_2^{m+1} = 2 \end{cases}$$
$\Rightarrow \boxed{a_2^{m+1} = b_2^{m+1}}$
Case 3.3:$a_2^m = 2$:
$$\overset{\mathrm{Def}}{\Rightarrow} \begin{cases} 1 = (m-2)(2t+2) \Rightarrow 2 = (m+1) - 2 \mod (2t+2) \Rightarrow a_2^{m+1} = 0 \\ a_2^m = 2 \Rightarrow b_2^{m+1} = 0 \end{cases}$$
$\Rightarrow \boxed{a_2^{m+1} = b_2^{m+1}}$

This concludes the induction step. □