Stability and nonlinear waves in damped driven rotating shallow water models

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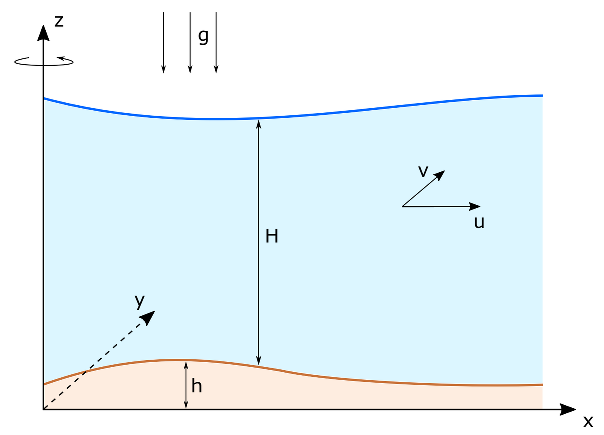
The inviscid rotating shallow water model

We consider the flow of an incompressible fluid with constant density in a flat and rotating layer as a simple model for the motions in the atmosphere and ocean.

If we start with the Euler equations and additionally presuppose very small characteristic fluid depth H_0 in comparison to the horizontal length scale as well as initial horizontal velocity which is independent in the vertical spatial direction, then we can derive the **inviscid rotating shallow water equations** (e.g. [1]):

$$(1)\quad \begin{cases} \partial_t u &= -u\partial_x u - v\partial_y u + fv - g\partial_x\eta \\ \partial_t v &= -u\partial_x v - v\partial_y v - fu - g\partial_y\eta \\ \partial_t\eta &= -\partial_x(u\eta) - \partial_y(v\eta) - H_0(\partial_x u + \partial_y v) \end{cases}$$

- $u(x,y,t)$, $v(x,y,t)$ – velocity component in the x-, y-direction
- $h(x,y)$ – bottom topography (here $h \equiv 0$)
- $H_0 > 0$ – characteristic fluid depth
- $H(x,y,t) = H_0 + \eta(x,y,t)$ – fluid depth
- $f \in \mathbb{R}\backslash\{0\}$ – Coriolis parameter
- $g > 0$ – gravitational parameter



Here we have no vertical velocity component and spatial direction anymore. That means the number of equations, dependent and independent variables are reduced by one.

We also consider an unbounded spatial domain $(x,y) \in \mathbb{R}^2$ and thus investigate motions and waves without the influence of boundaries on them.

Spectral stability analysis

We investigate the spectral stability of the trivial steady solutions of (1)

$$(u,v,\eta) \equiv (0,0,s) \text{ for any } s > -H_0.$$

Here the parameter s can be set to zero, because this parameter is equivalent to the change of the characteristic fluid depth $H_0 \to H_0 + s$, which is chosen arbitrarily here.

For the spectral stability we linearise equation (1) at the trivial solution and investigate the eigenvalues λ of the linearisation concerning the Fourier modes

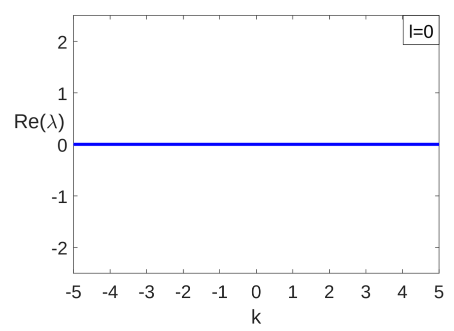
$$e^{i(kx+\ell y)+\lambda t} \cdot (\tilde{u},\tilde{v},\tilde{\eta}).$$

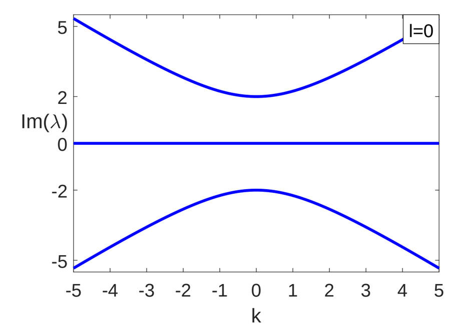
This leads to the dispersion relation

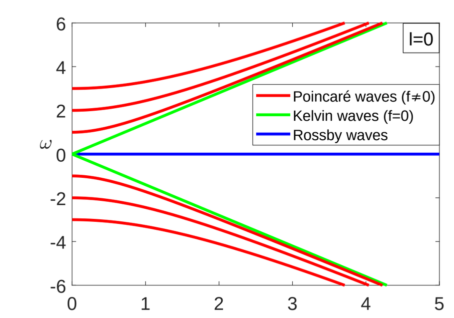
$$0 = \lambda(\lambda^2 + gH_0(k^2+\ell^2) + f^2).$$

Due to $\mathrm{Re}(\lambda) = 0$ for all $k,\ell \in \mathbb{R}$ the trivial steady solution is always **neutrally stable** (see top figure on the right).

The frequency $\omega := \mathrm{Im}(\lambda)$ describes different wave phenomena in the ocean. They are classified in **Rossby waves** ($\omega = 0$), **Poincaré waves** ($\omega = \sqrt{gH_0(k^2+\ell^2)+f^2}$) for $f \neq 0$, also known as **inertia-gravity waves**, and **Kelvin waves** ($\omega = \sqrt{gH_0}||(k,\ell)||$), which exist at the equator with $f = 0$. The wave types are depicted in the bottom figure on the right for $\ell = 0$ and different values of f. In the presence of a bottom slope the frequency of the Rossby waves is only zero for $k = \ell = 0$ and otherwise nonzero and very small.







Stability and bifurcations in damped driven rotating shallow water equations

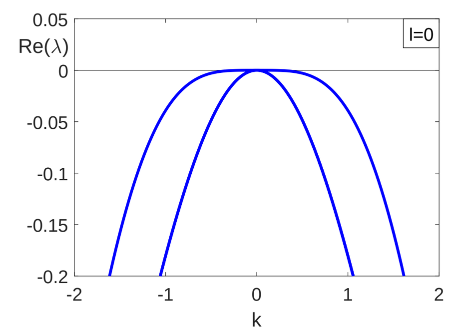
In geophysical fluid models we also have damping due to viscosity and driving for various reasons. Here we investigate the effects of different damping and driving mechanisms on the dynamics. One way to get a better understanding of the dynamics is to study stability changes and bifurcations of steady solutions.

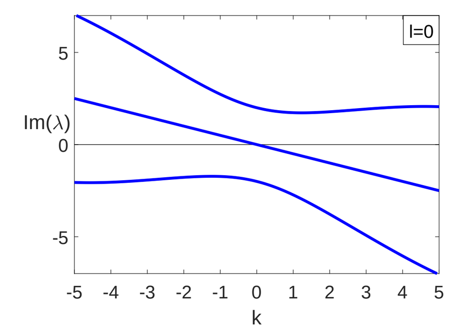
Wind stress

$$(2)\quad \begin{cases} \partial_t u &= -u\partial_x u - v\partial_y u + fv - g\partial_x\eta + d_1\Delta u + s_x \\ \partial_t v &= -u\partial_x v - v\partial_y v - fu - g\partial_y\eta + d_2\Delta v + s_y \\ \partial_t\eta &= -\partial_x(u\eta) - \partial_y(v\eta) - H_0(\partial_x u + \partial_y v) \end{cases}$$

with viscosity parameters $d_1, d_2 > 0$ and constant wind stress vector $(s_x, s_y) \in \mathbb{R}^2$.

- Often with nonconstant wind stress in the form of a cosine function in the first equation only, simulating wind in the zonal direction which is changing meridionally. This leads to quasi geostrophic double gyres (e.g. [2]).
- Trivial steady solutions $(u,v,\eta) = (s_y/f, -s_x/f, s)$ for any $s > -H_0$. Relative to the wind direction these steady motions go to the right in the northern hemisphere ($f > 0$) and to the left in the southern hemisphere ($f < 0$).
- The Laplace operators in (2) cause negative real part in the spectrum (see top figure on the right). Thus the trivial steady solutions are always **spectrally stable** and **marginally stable** in the origin of the Fourier space. The wind stress only changes the imaginary part of the eigenvalues and therefore has no influence on the stability.



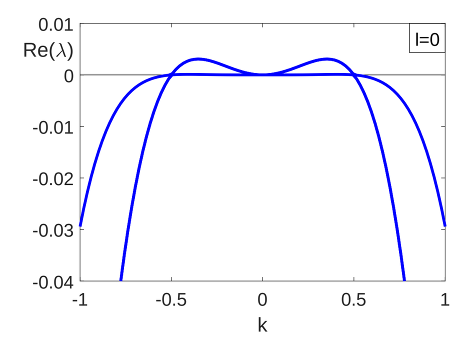


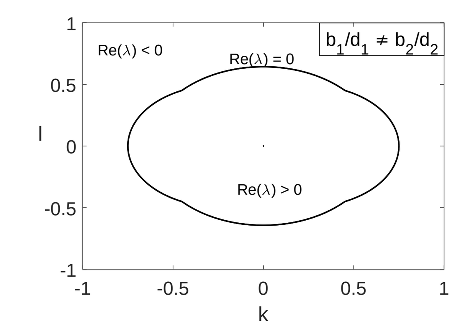
Backscatter

$$(3)\quad \begin{cases} \partial_t u &= -u\partial_x u - v\partial_y u + fv - g\partial_x\eta - d_1\Delta^2 u - b_1\Delta u \\ \partial_t v &= -u\partial_x v - v\partial_y v - fu - g\partial_y\eta - d_2\Delta^2 v - b_2\Delta v \\ \partial_t\eta &= -\partial_x(u\eta) - \partial_y(v\eta) - H_0(\partial_x u + \partial_y v) \end{cases}$$

with hyperviscosity parameters $d_1, d_2 > 0$ and backscatter parameters $b_1, b_2 > 0$.

- This model comes from the subgrid parametrisation and is intended to provide energy consistency in the simulations. In order to avoid the loss of too much eddy kinetic energy the negative Laplace operators re-inject it from the subgrid scale to the grid scale (e.g. [3]).
- Same trivial steady solutions $(u,v,\eta) = (0,0,s)$ for any $s > -H_0$. They are always **unstable** due to positive real part of the eigenvalues near the origin of the Fourier space (see top figure on the right).
- With the condition $b_1/d_1 = b_2/d_2$ the area of unstable waves is given by $0 < k^2+\ell^2 < b_1/d_1$ (the other case is depicted in the bottom figure on the right).



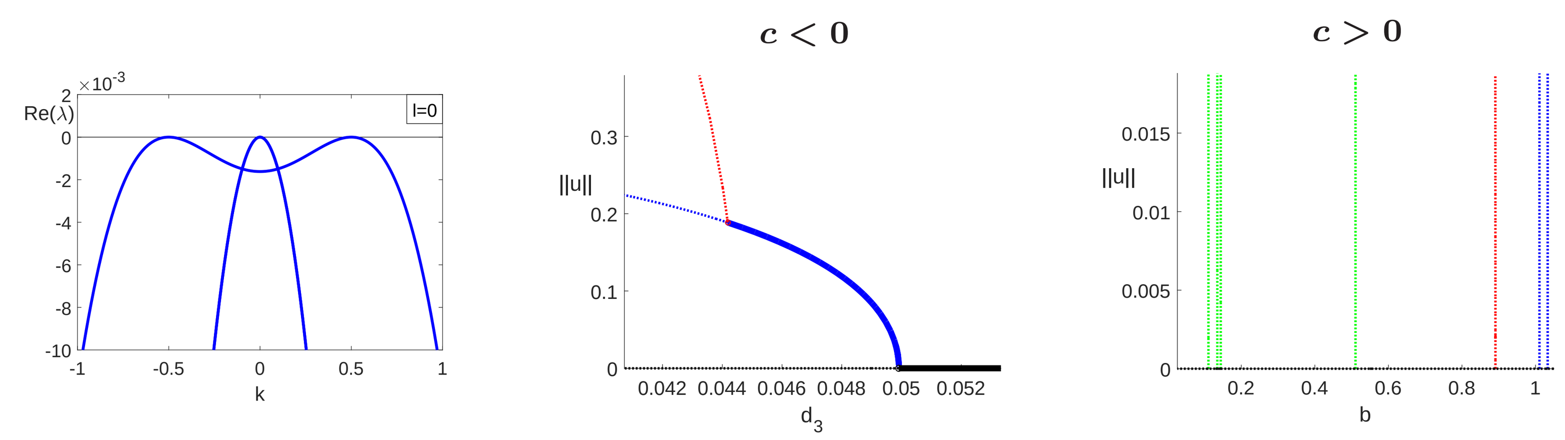


Mathematically motivated damping and driving

$$(4)\quad \begin{cases} \partial_t u &= -u\partial_x u - v\partial_y u + fv - g\partial_x\eta + d_1\Delta u + au \\ \partial_t v &= -u\partial_x v - v\partial_y v - fu - g\partial_y\eta + d_2\Delta v + bv \\ \partial_t\eta &= -\partial_x(u\eta) - \partial_y(v\eta) - H_0(\partial_x u + \partial_y v) + d_3\Delta\eta + c\eta \end{cases}$$

with damping parameters $d_1, d_2, d_3 > 0$ and $a, b, c \in \mathbb{R}$ (driving parameters if positive).

- Trivial steady solution $(u,v,\eta) = (0,0,0)$.
- For $c = 0$ similar stability behavior as in (2) and (3).
- For $c > 0$ always unstable due to positive real part of λ near $(k,\ell) = (0,0)$.
- For $c < 0$ stability changes are possible. Varying a damping parameter creates stability changes at the critical points outside the origin (see left figure below).
- Bifurcations for $c > 0$ exist, but they do not result from stability changes as for $c < 0$. It is possible to explain some of these vertical branches by exact nonlinear solutions.



Exact nonlinear solutions

Consider for $\zeta = kx + \ell y$, $(k,\ell) \in \mathbb{R}^2$, solutions of (1) of the form

$$u(x,y,t) = u(\zeta),\quad v(x,y,t) = v(\zeta),\quad \eta(x,y,t) = \eta(\zeta).$$

This leads to the system of ordinary differential equations

$$\begin{aligned} 0 &= -(ku+\ell v)u' + fv - gk\eta' \\ 0 &= -(ku+\ell v)v' - fu - g\ell\eta' \\ 0 &= -(ku+\ell v)\eta' - (ku+\ell v)'\eta - H_0(ku+\ell v)' \end{aligned}$$

- For $u(\zeta) = -\ell\psi(\zeta)$ and $v(\zeta) = k\psi(\zeta)$ the **nonlinear terms vanish**.
- $\psi(\zeta) = \frac{g}{f}\eta'(\zeta)$ solves the remaining linear problem with arbitrary function $\eta(\zeta)$.

Additional linear terms reduce the set of solutions. In (3) for instance: with the condition $b_1/d_1 = b_2/d_2$ we get the same solutions u and v by ψ with

$$\eta = \alpha \cdot sin(kx + \ell y + \beta) + s$$

for all $s, \alpha, \beta \in \mathbb{R}$ and $k, \ell \in \mathbb{R}$ with $k^2 + \ell^2 = b_1/d_1$. The same set of solutions, and also increasing/decreasing solutions in time, exist in (4) under different conditions. The arbitrary choice of the amplitude $\alpha \in \mathbb{R}$ is the reason for the vertical branches.

References

[1] Joseph Pedlosky: "Geophysical Fluid Dynamics", second edition, 1987, Springer-Verlag, New York

[2] Henk A. Dijkstra: "Nonlinear Physical Oceanography", second revised and enlarged edition, 2005, Springer, Dordrecht

[3] Malte F. Jansen, Isaac M. Held: "Parameterizing subgrid-scale eddy effects using energetically consistent backscatter", Ocean Modelling, Vol. 80, 2014, pp. 36-48

[4] Artur Prugger, Jens D. M. Rademacher: "Stability and explicit waves in damped driven shallow water models", in preparation, 2019