Fully Dynamic Algorithms for Knapsack Problems with Polylogarithmic Update Time

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July 20, 2020

Abstract

Knapsack problems are among the most fundamental problems in optimization. In the MULTIPLE KNAPSACK problem, we are given multiple knapsacks with different capacities and items with values and sizes. The task is to find a subset of items of maximum total value that can be packed into the knapsacks without exceeding the capacities. We investigate this problem and special cases thereof in the context of *dynamic algorithms* and design data structures that efficiently maintain near-optimal knapsack solutions for dynamically changing input. More precisely, we handle the arrival and departure of individual items or knapsacks during the execution of the algorithm with worst-case update time polylogarithmic in the number of items. As the optimal and any approximate solution may change drastically, we only maintain implicit solutions and support certain queries in polylogarithmic time, such as the packing of an item and the solution value.

While dynamic algorithms are well-studied in the context of graph problems, there is hardly any work on packing problems and generally much less on non-graph problems. Given the theoretical interest in knapsack problems and their practical relevance, it is somewhat surprising that KNAPSACK has not been addressed before in the context of dynamic algorithms and our work bridges this gap.

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1 Introduction

Knapsack problems are among the most fundamental optimization problems, studied since the early days of optimization theory. In its most basic form, there is given a knapsack with capacity $S \in \mathbb{N}$ and a set of n items, where each item $j \in [n] := \{1, 2, \ldots, n\}$ has a size $s_j \in \mathbb{N}$ and a value $v_j \in \mathbb{N}$. The KNAPSACK problem asks for a subset of items, $P \subseteq [n]$, with maximal total value $v(P) := \sum_{j \in P} v_j$ and with a total size $s(P) := \sum_{j \in P} s_j$ that does not exceed the knapsack capacity S. In the more general MULTIPLE KNAPSACK problem, we are given m knapsacks with capacities S_i for $i \in [m]$. Here, the task is to select m disjoint subsets $P_1, P_2, \ldots, P_m \subseteq [n]$ such that subset P_i satisfies the capacity constraint $s(P_i) \leq S_i$ and the total value of all subsets $\sum_{i \in [m]} v(P_i)$ is maximized.

The decision variant of MULTIPLE KNAPSACK is strongly NP-complete, even for identical knapsack capacities, as it is a special case of bin packing. The KNAPSACK problem, on the other hand, is only weakly NP-complete in its decision variant – in fact, it is one of the 21 problems on Karp's list of NP-complete problems [49] – and it admits pseudo-polynomial time algorithms [5]. The first published pseudo-polynomial time algorithm for KNAPSACK from the 1950s has run time $\mathcal{O}(n \cdot S)$ [5].

As a consequence of these hardness results, each of the knapsack variants has been studied extensively over the years through the lens of approximation algorithms. Of particular interest are *approximation schemes*, families of polynomial-time algorithms that compute for each $\varepsilon > 0$ a $(1-\varepsilon)$-approximate solution, i.e., a feasible solution with value within a factor of $(1-\varepsilon)$ of the optimal solution value (see also related work). The first approximation scheme for the KNAPSACK problem is due to Ibarra and Kim [38] and initiated a long sequence of follow-up work, with the latest improvements appearing only recently [17, 47].

MULTIPLE KNAPSACK is substantially harder and does not admit $(1-\varepsilon)$-approximate algorithms with running time polynomial in $\frac{1}{\varepsilon}$ unless P = NP, even with two identical knapsacks [18]. However, some approximations schemes with exponential dependency on $\frac{1}{\varepsilon}$ are known [18, 50] as well as improved variants where the dependency on $f\left(\frac{1}{\varepsilon}\right)$ for some function f is only multiplicative or additive [43, 45]. The currently fastest known algorithm has a runtime of $2^{\mathcal{O}(\log^4(1/\varepsilon)/\varepsilon)} + \mathrm{poly}(n)$ [45]. All these algorithms are *static* in the sense that the full instance is given to an algorithm and is then solved.

The importance for theory and practice is reflected by the two books on knapsack problems [52, 61]. Given the relevance of knapsack applications in practice and the ubiquitous dynamics of real-world instances, it is natural to ask for *dynamic algorithms* that adapt to small changes in the packing instance while spending only little computation time. More precisely, during the execution of the algorithm, items and knapsacks arrive and depart and the algorithm needs to maintain an approximate knapsack solution with an *update time* polylogarithmic in the number of items in each step. A dynamic algorithm is then a data structure that implements these updates efficiently and supports relevant query operations. To the best of our knowledge, we are the first to analyze knapsack problems in the context of dynamic algorithms.

Generally, dynamic algorithms constitute a vibrant research field in the context of graph problems. Refer to surveys [15, 24, 34] for an overview on dynamic graph algorithms. Interestingly, only for a small number of graph problems there are dynamic algorithms known with *polylogarithmic* update time, among them connectivity problems [35, 37], the minimum spanning tree [37], and vertex cover [10, 12]. Recently, this was complemented by conditional lower bounds that are typically *linear* in the number of nodes or edges; see, e.g., [2]. Over the last few years, the generalization of dynamic vertex cover to dynamic set cover gained interest leading to near-optimal approximation algorithms with polylogarithmic update times [1, 9, 11, 30].

For packing and, generally, for non-graph-related problems, dynamic algorithms with small update time are much less studied. A notable exception is a result for bin packing that maintains a $\frac{5}{4}$-approximative solution with $\mathcal{O}(\log n)$ update time [40]. This lack of efficient dynamic algorithms is in stark contrast to the aforementioned intensive research on computationally efficient algorithms for knapsack problems.

Besides the purely theoretical point of view, packing questions appear in many applications and efficient algorithms for answering these questions are highly relevant in practice. One such example is estimating the profit of scheduling jobs onto several server hosting centers. Cloud providers need to adhere to certain Service Level Agreements while efficiently managing their resources. In recent years, it has become apparent that the cost of powering large-scale computing infrastructures surpasses the hardware cost after only few years. Thus, the task of dynamically migrating workload as virtual machines between several computing clusters has evolved [6]. This allows the service provider to adapt the provided capacity, i.e., the currently running computing clusters, to the current demand. Situations of this flavor are addressed in [13, 21, 58].

A framework for MULTIPLE KNAPSACK with efficient update times can be viewed as a first-stage decision tool: In real-time, it can be determined whether the customer in question should be allowed into the system based on the cost of potentially powering and using additional servers. As the service provider has to decide immediately which request she wants to accept, she needs to obtain the information *fast*, i.e., sublinear in the number of requests already in the system.

Given the theoretical interest in knapsack problems and their practical relevance, it is surprising that KNAPSACK has not been addressed in the context of dynamic algorithms. Our work bridges this gap initiating the design of data structures and algorithms that efficiently maintain near-optimal knapsack solutions.

Our Contribution

In this paper, we present dynamic algorithms for maintaining approximate knapsack solutions for three problems of increasing complexity: KNAPSACK, MULTIPLE KNAPSACK with identical knapsack sizes, and MULTIPLE KNAPSACK. Our algorithms are *fully dynamic* which means that in an update operation they can handle both, the arrival or departure of an item and the arrival or departure of a knapsack. Further, we consider the *implicit solution* or *query* model, in which an algorithm is not required to store the solution explicitly in memory such that the solution can be read in linear time at any given point of the execution. Instead, the algorithm may maintain the solution implicitly with the guarantee that a query about the packing can be extracted in polylogarithmic time. Moreover, since KNAPSACK is already NP-hard even with full knowledge of the instance, we aim at maintaining $(1-\varepsilon)$-approximate solutions.

We give *worst-case* guarantees for update and query times that are polylogarithmic in n, the number of items currently in the input, and bounded by a function of $\varepsilon > 0$, the desired approximation accuracy. For some special cases, we can even ensure a polynomial dependency on $\frac{1}{\varepsilon}$. In others, we justify the exponential dependency with NP-hardness results. Denote by $v_{\max}$ the currently largest item value and by $\overline{v}$ an upper bound on $v_{\max}$ that is know in advance. Let $S_{\max}$ be the currently largest knapsack capacity.

- For MULTIPLE KNAPSACK, we design a dynamic algorithm maintaining a $(1-\varepsilon)$-approximate solution with update time $\left(\frac{1}{\varepsilon}\log n \log \overline{v}\right)^{\mathcal{O}(1/\varepsilon)}$ and query time $\mathcal{O}(\frac{\log n}{\varepsilon^2})$ for single items. (Theorem 6.1)
- The exponential dependency on $\frac{1}{\varepsilon}$ in the update time for MULTIPLE KNAPSACK is indeed necessary, even for two identical knapsacks. We show that there is no $(1-\varepsilon)$-approximate dynamic algorithm with update time $\left(\frac{1}{\varepsilon}\log n\right)^{\mathcal{O}(1)}$, unless P = NP. (Theorem 3.3)
- For KNAPSACK, we give a $(1-\varepsilon)$-approximation algorithm with update time $\mathcal{O}\left(\frac{\log^4(n v_{\max})}{\varepsilon^9}\right) + \mathcal{O}(\frac{1}{\varepsilon}\log n \log \overline{v})$ and constant query times. (Theorem 4.1)
- For MULTIPLE KNAPSACK with m *identical knapsacks*, we maintain a $(1-\varepsilon)$-approximate solution with update time $\left(\frac{1}{\varepsilon}\log n \log v_{\max} \log S_{max}\right)^{\mathcal{O}(1)}$ and query time $\left(\frac{1}{\varepsilon}\log n\right)^{\mathcal{O}(1)}$ if $m \geq \frac{16}{\varepsilon^7}\log^2 n$. For small m, we get an exponential dependency on $\frac{1}{\varepsilon}$ by extending the result for KNAPSACK.

In each update step, we compute only implicit solutions and provide queries for the solution value, the knapsack of a queried item, or the complete solution. These queries are consistent between two update steps and run efficiently, i.e., polynomial in $\log n$ and $\log \overline{v}$ and with a dependence on ε and the output size. We remark that it is not possible to maintain a solution with a non-trivial approximation guarantee explicitly with only polylogarithmic update time (even amortized) since it might be necessary to change $\Omega(n)$ items per iteration, e.g., if a very large and very profitable item is inserted and removed in each iteration.

Related Work

Since the first approximation scheme[1] for KNAPSACK by Ibarra and Kim [38], running times have been improved steadily [17, 26, 27, 47, 51, 57, 68] with $\mathcal{O}(n \log \frac{1}{\varepsilon} + (\frac{1}{\varepsilon})^{9/4})$ by Jin [47] being the currently fastest. Recent work on conditional lower bounds [20, 56] implies that KNAPSACK does not admit an FPTAS with runtime $\mathcal{O}((n + \frac{1}{\varepsilon})^{2-\delta})$, for any $\delta > 0$, unless $(min, +)$-convolution has a subquadratic algorithm [17, 64].

A PTAS for MULTIPLE KNAPSACK was first discovered by Chekuri and Khanna [18] and an EPTAS due to Jansen [43] is also known. The running time of the EPTAS is $2^{\mathcal{O}(\log(1/\varepsilon)/\varepsilon^5)} \cdot \text{poly}(n)$. Jansen later presented a second EPTAS [45] with an improved running time of $2^{\mathcal{O}(\log^4(1/\varepsilon)/\varepsilon)} + \text{poly}(n)$. The mentioned algorithms are all static and assume full knowledge about the instance for which a complete solution has to be found. Hence, directly applying such an approximation scheme on each update is prohibitive as a single item arrival can change a packing solution completely requiring a full recomputation with running time polynomial in the input size.

At the heart of the two EPTASes [43, 45] lies a configuration integer linear program (ILP) for rounded items and/or knapsacks of exponential size. Even though near-optimal solutions to the LP relaxations can be found and rounded in time $\mathcal{O}(\text{poly}(n))$, this is beyond the scope of the polylogarithmic update time we are interested in. Additionally, the configuration ILPs still contain $\mathcal{O}(n)$ many constraints and variables which is yet another obstacle when aiming for dynamically maintaining approximate solutions with polylogarithmic running time. Hence, to sufficiently improve the running time, a more careful approach for rounding items has to be developed before similar configuration ILPs can be applied.

The dynamic arrival and removal of items exhibits some similarity to knapsack models with incomplete information. Somewhat related to our work are models in which the set of items that has to be packed is not fully known. In the *online* knapsack problem [60] items arrive online one by one. When an item arrives, an algorithm must accept or reject it before the next item arrives, and this decision is irrevocable. The goal is to maintain a knapsack solution that has a value close to the optimal value achievable when all items are known in advance. Contrary to our model, the available computational power is assumed to be unlimited (though many known algorithms run in polynomial time) as the key difficulty lies in making irrevocable packing decisions under uncertainty. Various problem variants have been studied, e.g., online knapsack with resource augmentation [42], the removable online knapsack problem [19, 31–33, 41], in which items can be discarded later, the online partially fractional knapsack problem [65], items arriving in a random order [3, 4], the stochastic online knapsack problem [53, 54, 60, 72], and online knapsack with advice [14].

Other models that take into account uncertainty in the item set or in the available knapsack capacity include the *stochastic* knapsack problem [8, 23, 59] and various *robust* knapsack problems [16, 25, 62, 74]. While some models allow for an adjustment of the solution after the realization of a scenario, the major focus lies on constructing a new packing of high value instead of the computational complexity of the update or the data structures necessary to maintain (approximately) optimal packings.

Finally, we mention two other seemingly related research streams towards more adaptive online models with a softened irrevocability requirement. Online optimization with *recourse* [29, 39, 63] or *migration* [46, 70, 71] allows to adapt previously taken decisions in a limited way. We are no aware of work on knapsack problems and, again, the goal is to bound the amount of change needed to maintain good online solutions regardless of the computational effort.

[1] An approximation scheme is a family of polynomial-time algorithms computing a $(1-\varepsilon)$-approximation for every $\varepsilon > 0$. Based on the dependency on ε of the respective running time, we distinguish *Polynomial Time Approximation Schemes (PTAS)* with arbitrary dependency on ε, *EPTAS* where arbitrary functions $f(\varepsilon)$ may only appear as a multiplicative factor, and *FPTAS* with polynomial dependency on ε.

2 Methodology and Roadmap

This section serves as a blueprint for our design of a dynamic algorithm for MULTIPLE KNAPSACK providing a high-level overview over the key components and technical challenges. While the improved algorithms for some special cases have significance on their own, the used techniques further complement each other and facilitate our main goal of handling arbitrary instances, which we discuss in Section 6.

We accomplish this goal by partitioning the given knapsacks based on their respective capacity, creating two subproblems of MULTIPLE KNAPSACK. This separation allows us to design algorithms that exploit the structural properties specific to the respective problem. One subproblem consists of relatively few knapsacks, but they are the largest of the instance. While the small number of these *special* knapsacks offers more algorithmic freedom, this freedom is necessary since great care has to be taken when computing a solution. After all, there may be items of high value that only fit into special knapsacks. The second subproblem contains almost all of the remaining smaller knapsacks. The sheer number of these *ordinary* knapsacks results in an inverse problem, with the algorithmic handling of the numerous knapsacks being a major hurdle. On the upside, mistakes are forgiven more easily allowing us to even discard a small fraction of knapsacks entirely. Additionally, we create a third partition of knapsacks that lies in-between the two subproblems (w.r.t. knapsack capacity). It consists of knapsacks that contribute negligible value to an optimal solution. This property induces the precise partitioning and allows us to consider the knapsacks as empty *extra* knapsacks, which we use to place leftover items not packed in the subproblems.

The major challenge with this divide-and-conquer approach is to decide which item is assigned to which of the two subproblems. Clearly, for some – *special* – items this question is answered by their size as they only fit into special knapsacks, unlike the remaining – *ordinary* – items. In fact, for them the allocation is so problematic that we resort to downright putting a number of high-value ordinary items into extra knapsacks. To handle the remainder, we guess the total size of ordinary items that are put into special knapsacks by an optimal solution. We then add a virtual knapsack – with capacity equal to this guess – to the ordinary subproblem and solve it with the not yet packed ordinary items as input. The input for the special subproblem then consists of all special items together with bundles of the ordinary items packed in the virtual knapsack.

Special Knapsacks: The approach for the few (m_S) special knapsacks can be divided itself into two parts that consider high- and low-value items respectively. The corresponding partition is guessed so that the high-value items contain the $\frac{m_S}{\varepsilon^2}$ most valuable items of an optimal solution, denoted by OPT, and the low-value items the remaining items of OPT. For the important high-value items, a good solution is paramount, so we employ an EPTAS for MULTIPLE KNAPSACK. It is run on a low-cardinality set of high-value candidate items together with $\frac{m_S}{\varepsilon}$ placeholders of equal size that reserve space for low-value items. The values of placeholders are determined by filling them fractionally with the densest low-value items. We aim to reserve a total space equal to that of all low-value items in OPT, which we guess up to a factor of $(1+\varepsilon)$. Thanks to the partitioning, we can charge cut items to the high-value items in OPT. This yields an update time $2^{f(1/\varepsilon)}(\frac{m}{\varepsilon}\log(nv_{\max}))^{\mathcal{O}(1)} + \mathcal{O}(\frac{1}{\varepsilon}\log\overline{v}\log n)$, with f quasi-linear, query time $\mathcal{O}(\log\frac{m^2}{\varepsilon^6})$ for individual items and time $\mathcal{O}(|P|)$ to output the solution P for all or a single knapsack. See Appendix B for the full description and analysis of this algorithm.

Ordinary Knapsacks: Having the extra knapsacks available for resource augmentation enables us to set up a configuration linear program (LP) for the ordinary knapsacks and place any fractional item or configuration into the extra knapsacks. To solve the LP efficiently and achieve an update time of $\left(\frac{1}{\varepsilon}\log n\right)^{\mathcal{O}(1/\varepsilon)}(\log m \log S_{\max} \log v_{\max})^{\mathcal{O}(1)}$ we significantly decrease the number of ordinary items with a new, dynamic approach to linear grouping.

Dynamic linear grouping clusters a (sub)set of items into so-called *item types* of roughly the same size and value in time $\left(\frac{1}{\varepsilon}\log n\right)^{\mathcal{O}(1)}$. Traditionally, linear grouping is applied in bin packing problems where any feasible solution contains all items [22]. This property is crucial since the cardinality of the groups depends on the number of packed items. In knapsack problems, however, a feasible solution may consist of only a subset of items. We handle this uncertainty by simultaneously executing classical linear grouping for $\mathcal{O}(\log_{1+\varepsilon} n)$ many guesses of its cardinality and, thus, simulate the possible choices of an optimum.

We call an item type *small* or *big* with respect to a particular knapsack if its size is at most or at least an ε-fraction of the knapsack's capacity, respectively. Based on this, we partition ordinary knapsacks further into groups such that the classification into small, big or "does not fit" is consistent among knapsacks of a given group. Note that these groups are not based on m and $\frac{1}{\varepsilon}$ but dynamically depend on the current set of items. As the number of big items per knapsack is bounded, the LP explicitly assigns those via configurations to knapsacks while it assumes that small items can be packed fractionally and, thus, those are only assigned by number to the respective groups. This algorithm is described and analyzed in Appendix E.

To highlight major algorithmic ideas and technical contributions, we present algorithms for the following more accessible special cases, which can – with some additional effort – be generalized to the subproblems discussed above. Beyond that, they are of independent interest as they constitute relevant special cases of MULTIPLE KNAPSACK and allow for significant improvements in the running time.

Single Knapsack: When the instance consists of only a single knapsack, the approach used for special knapsacks allows for greatly improved update and query times. For KNAPSACK, we can replace the EPTAS by a faster FPTAS and, additionally, we need only a single placeholder item. The remaining steps are unchanged, yielding an update time of $\mathcal{O}\big(\frac{\log^4 n v_{\max}}{\varepsilon^9}\big) + \mathcal{O}(\frac{1}{\varepsilon}\log n \log \overline{v})$. Moreover, the single placeholder enables us to compute its membership by comparing with a saved pivot element. Thus, we can access any item in constant time and de facto compute explicit solutions. We present these results in Section 4.

Identical Knapsacks: As a special case of the second subproblem, we consider MULTIPLE KNAPSACK with identical capacities, where an item is either big or small with respect to all knapsacks. As a consequence, (i) this exponentially decreases the number of constraints, and (ii) it suffices to reserve some capacity of the knapsacks for packing the densest small items instead of computing their exact number by the configuration LP. Still, the number of variables is prohibitively large. Hence, we would like to apply the Ellipsoid method with an approximate separation oracle to the dual problem similar to [48, 67, 69]. However, we cannot use their approaches directly due to two additional variables in the dual problem. Instead, we add an objective function constraint to the dual and carefully exploit the connection between feasible and infeasible dual solutions to obtain a basic feasible solution for the primal. This enables us to approximately solve the LP and round a solution in time $\big(\frac{1}{\varepsilon}\log n \log v_{\max} \log S_{max}\big)^{\mathcal{O}(1)}$ if m is sufficiently large.

3 Data Structures and Preliminaries

From the perspective of a data structure that implicitly maintains near-optimal solutions for MULTIPLE KNAPSACK, our algorithms support several different update and query operations. These allow for the input to MULTIPLE KNAPSACK to be changed, which causes the computation of a new solution, or for (parts of) that solution to be output, respectively. The supported update operations are as follows.

- **Insert Item:** inserts an item into the input
- **Remove Item j:** removes item j from the input
- **Insert Knapsack:** inserts a knapsack into the input
- **Remove Knapsack i:** removes knapsack i from the input

These compute a new solution which can be output, entirely or in parts, using the following query operations.

- **Query Item j:** returns whether item j is packed in the current solution and if yes, additionally returns the knapsack containing it
- **Query Solution Value:** returns the value of the current solution

- **Query Entire Solution:** returns all items in the current solution, together with the information in which knapsack each such item was packed

In particular, these queries are consistent in-between two update operations. Nevertheless, the answers to queries are not independent of each other but depend on the precise queries as well as their order.

To provide the above functionality, we require the use of additional auxiliary data structures and make a few basic assumptions which we now discuss. First, while the model imposes no time bounds on the computation of an initial solution, it can be easily seen that for all our algorithms such an initial solution can be computed in time nearly linear in n and with additional dependencies on ε, $v_{\max}$, and $\overline{v}$ as in the respective algorithms. For simplicity, we assume that elementary operations such as addition, multiplication, and comparison of two values can be handled in constant time. Clearly, this is not true as the parameters involved can be as large as $v_{\max}$ and $S_{\max} := \max S_i$. However, as the number of elementary operations is bounded, their results do not grow arbitrarily large but are in fact bounded by a polynomial in $\log n$, $\log m$, $S_{\max}$, and $v_{\max}$ and some function of $\frac{1}{\varepsilon}$. Thus, we do not explicitly state the size of the involved numbers. Lastly, we make following standard assumptions on ε. By appropriately decreasing ε, we assume without loss of generality that $\frac{1}{\varepsilon} \in \mathbb{N}$. If m is sufficiently large, i.e., $m \geq \frac{1}{\varepsilon}$, we also assume $\varepsilon m \in \mathbb{N}$.

Rounding Values A crucial ingredient to our algorithms is the partitioning of items into only few *value classes* V_ℓ consisting of items j for which $(1+\varepsilon)^\ell \leq v_j < (1+\varepsilon)^{\ell+1}$. Upon arrival of an item, we calculate its value class V_{ℓ_j} and store j together with v_j, s_j, and ℓ_j in the appropriate data structures of the respective algorithm. We assume all items in V_ℓ to have value $(1+\varepsilon)^\ell$ and, abusing notation, use V_ℓ to refer to both the value class and the (rounded) value of its items. The following lemma justifies this assumption. Since this technique is rather standard, we only state the lemma without providing a formal proof.

Lemma 3.1. *(i) There are at most* $\mathcal{O}(\frac{\log v_{\max}}{\varepsilon})$ *many value classes.*
(ii) For optimal solutions $\textsc{Opt}$ *and* $\textsc{Opt}'$ *for the original and rounded instance respectively, we have* $v(\textsc{Opt}') \geq (1-\varepsilon) \cdot v(\textsc{Opt})$.

Data Structures The targeted running times do not allow for completely reading the instance in every round but rather ask for carefully maintained data structures that allow us to quickly compute and store implicit solutions. For access to the input, we maintain an array that for each item stores the item's size, value and value class, and similarly for knapsacks. Mainly, however, our dynamic algorithms rely on maintaining sorted lists of up to n or m items or knapsacks respectively. For all sortings, break ties according to indices.

As our goal is to design algorithms with poly-logarithmic update times, it is crucial that the data structures enable accordingly efficient insertion, deletion and access times. Bayer and McCreight developed such a data structure in 1972, the so-called B-trees that were later refined by Bayer to *symmetric binary* B*-trees*. In contrast to this early work, we additionally store in each node k information such as the total size, the total value, the total number of elements or the total capacity of the subtree rooted in k.

As observed by Olivié [66] and by Tarjan [73], updating the original symmetric binary B-trees can be done with a constant number of rotations. For our dynamic variant of B-trees, this implies that only a constant number of internal nodes are involved in an update procedure. In particular, if a subtree is removed or appended to a certain node, only the values of this node and of his predecessors need to be updated. The number of predecessors is bounded by the height of the tree which is logarithmic in the number of its leaves. Hence, the additional values stored in internal nodes can be maintained in time $\mathcal{O}(\log n)$ or $\mathcal{O}(\log m)$. Storing the additional values such as total size of a subtree in its root allows us to compute prefixes or the prefix sum with respect to these values in time $\mathcal{O}(\log n')$ as well. *Prefix computation* refers to finding the maximal prefix of the sorted list such that the elements belonging to the prefix have values whose sum is bounded by a given input. We return a prefix by outputting the index of its last element.

Lemma 3.2. *There is a data structure maintaining a sorting of n' elements w.r.t. to key value. Moreover,*

(i) insertion, deletion, or search by key value or index of an element takes time $\mathcal{O}(\log n')$, and

(ii) prefixes and prefix sums with respect to additionally stored values can be computed in time $\mathcal{O}(\log n')$.

Hardness of Computation To conclude this section, we provide a justification for the different running times of our algorithms for MULTIPLE KNAPSACK depending on the number of knapsacks. It is known that MULTIPLE KNAPSACK with $m = 2$ does not admit an FPTAS, unless P = NP [18]. For the dynamic setting, this implies that there is no dynamic algorithm with running time polynomial in $\log n$ and $\frac{1}{\varepsilon}$ unless P = NP. We show that a $(1-\varepsilon)$-approximate dynamic algorithm for MULTIPLE KNAPSACK with $m < \frac{1}{3\varepsilon}$ with update time polynomial in $\log n$ and $\frac{1}{\varepsilon}$ would imply that 3-PARTITION can be decided in polynomial time. The proof of following theorem is given in Appendix G.1. Note that this result can be extended to a larger number of knapsacks by adding an appropriate number of sufficiently small knapsacks.

Theorem 3.3. *Unless* P = NP*, there is no fully dynamic algorithm for* MULTIPLE KNAPSACK *that maintains a $(1-\varepsilon)$-approximate solution in update time polynomial in $\log n$ and $\frac{1}{\varepsilon}$, for $m < \frac{1}{3\varepsilon}$.*

4 A Single Knapsack

The first problem we consider is KNAPSACK. We show how to take advantage of dealing with only a single knapsack when maintaining $(1-\varepsilon)$-approximate solutions. Utilizing an FPTAS to pack a low-cardinality set of high-value candidates as well as a placeholder for low-value items, we obtain the following result.

Theorem 4.1. *For $\varepsilon > 0$, there is a fully dynamic algorithm for* KNAPSACK *that maintains $(1-\varepsilon)$-approximate solutions with update time $\mathcal{O}\big(\frac{\log^4(nv_{\max})}{\varepsilon^9}\big) + \mathcal{O}(\frac{1}{\varepsilon}\log n \log \overline{v})$. Furthermore, single items and the solution value can be accessed in time $\mathcal{O}(1)$.*

Definitions and Data Structures Denote by OPT the item set of an optimal solution and by $\text{OPT}_{\frac{1}{\varepsilon}}$ the $\frac{1}{\varepsilon}$ most valuable items of OPT. In both cases, break ties by picking smaller items. Denote by $V_{\ell_{\max}}$ and $V_{\ell_{\min}}$ the highest resp. lowest value (class) of an element in $\text{OPT}_{\frac{1}{\varepsilon}}$ and let $n_{\min} := \big|\text{OPT}_{\frac{1}{\varepsilon}} \cap V_{\ell_{\min}}\big| \le \frac{1}{\varepsilon}$. Furthermore, denote by $\mathcal{V}_L$ the value of the items in $\text{OPT} \setminus \text{OPT}_{\frac{1}{\varepsilon}}$, rounded down to a power of $(1+\varepsilon)$.

To efficiently run our algorithm we maintain several of the data structures from Section 3. We store the items of each non-empty value class V_ℓ (at most $\log_{1+\varepsilon} v_{\max}$) in a data structure ordered by increasing size. Second, for each possible value class V_ℓ (at most $\log_{1+\varepsilon} \overline{v}$), we maintain a data structure ordered by decreasing density that contains all items of value V_ℓ or lower. In particular, we maintain such a data structure even if V_ℓ is empty since initialization is prohibitively expensive in terms of run time. Instead we constantly maintain all data structures leading to the additive term in the update time of $\mathcal{O}(\log n \log_{1+\varepsilon} \overline{v})$. We use additional data structures to store our solution and support queries described in the proof of Lemma 4.2.

Algorithm The algorithm computes an implicit solution as follows.

1) **Compute a set $H_{\frac{1}{\varepsilon}}$ of high-value candidates:** Guess $V_{\ell_{\max}}$, $V_{\ell_{\min}}$ and $n_{\min}$. If $V_{\ell_{\min}} \ge \varepsilon^2 \cdot V_{\ell_{\max}}$, define $H_{\frac{1}{\varepsilon}}$ to be the set containing the $\frac{1}{\varepsilon}$ smallest items of each of the value classes $V_{\ell_{\min}+1}, \dots, V_{\ell_{\max}}$, plus the $n_{\min}$ smallest items from $V_{\ell_{\min}}$. Otherwise, set $H_{\frac{1}{\varepsilon}}$ to be the union of the $\frac{1}{\varepsilon}$ smallest items of each of the value classes with values in $[\varepsilon^2 \cdot V_{\ell_{\max}}, V_{\ell_{\max}}]$.

2) **Create a placeholder item B:** Guess $\mathcal{V}_L$ and consider the data structure of items with value at most $V_{\ell_{\min}}$ sorted by decreasing density. Remove the $n_{\min}$ smallest items of $V_{\ell_{\min}}$ until the next update. For the remaining items, compute the minimal size of fractional items necessary to reach a value $\mathcal{V}_L$. Then B is given by $v_B = \mathcal{V}_L$ and with s_B equal to the size of those low-value items.

3) **Use an FPTAS:** On the instance I, consisting of $H_{\frac{1}{\varepsilon}}$ and the placeholder item B, run an FPTAS parameterized by ε (we use the one by Jin [47]) to obtain a packing P.

4) **Implicit solution:** Among all guesses, keep the solution P with the highest value. Pack items from $H_{\frac{1}{\varepsilon}}$ as in P and, if $B \in P$, also pack the low-value items completely contained in B. While used candidates can be stored explicitly, low-value items are given only implicitly by saving the correct guesses and computing membership in B on a query.

Analysis. The above algorithm attains an approximation ratio of $(1-4\varepsilon) \cdot v(\textsc{Opt})$. One ε-fraction of $v(\textsc{Opt})$ is lost by using the FPTAS and additional one in each of the following three places. To obtain a candidate set $H_{\frac{1}{\varepsilon}}$ of constant cardinality, we restrict item values to $[\varepsilon^2 \cdot V_{\ell_{\max}}, V_{\ell_{\max}}]$. Since $|\textsc{Opt}_{\frac{1}{\varepsilon}}| = \frac{1}{\varepsilon}$, this removes items with a total value of at most $\frac{1}{\varepsilon} \cdot \varepsilon^2 V_{\ell_{\max}} \leq \varepsilon \cdot \textsc{Opt}$. Furthermore, due to guessing $\mathcal{V}_L$ up to a power of $(1+\varepsilon)$, we get $v_B = \mathcal{V}_L \geq (1-\varepsilon) \cdot v(\textsc{Opt} \setminus \textsc{Opt}_{\frac{1}{\varepsilon}})$. Finally, the item cut fractionally in Step 2) is charged to the $\frac{1}{\varepsilon}$ items of $\textsc{Opt}_{\frac{1}{\varepsilon}}$ which have a larger value.

The running time can be verified easily by multiplying the numbers of guesses for each value as well as the running time of the FTPAS. The latter is $\mathcal{O}(\frac{1}{\varepsilon^4})$, since we designed $H_{\frac{1}{\varepsilon}}$ to contain only a constant number of items, namely $\mathcal{O}(\frac{1}{\varepsilon^3})$ many. For a detailed analysis, see Appendix A.

Queries We show how to efficiently handle the different types of queries and state their runtime.

- **Single Item Query:** If the queried item is contained in $H_{\frac{1}{\varepsilon}}$, its packing was saved explicitly. Otherwise, if B is packed, we save the last – i.e., least dense – item contained entirely in B. By comparing with this item, membership in B can be decided in constant time on a query.
- **Solution Value Query:** While the algorithm works with rounded values, we may set up the data structure of Section 3 to additionally store the actual values of items. We store the actual solution value in the update step by adding the actual values of packed candidates and determining the actual value of items in B with a prefix computation. On query, we return the stored solution value.
- **Query Entire Solution:** Output the stored packing of candidates. If B was packed, iterate over items in B in the respective density-sorted data structure and output them.

Lemma 4.2. *The query times of our algorithm are as follows.*
(i) Single item queries are answered in time $\mathcal{O}(1)$.
(ii) Solution value queries are answered in time $\mathcal{O}(1)$.
(iii) Queries of the entire solution P are answered in time $\mathcal{O}(|P|)$.

5 Identical Knapsacks

5.1 Dynamic Linear Grouping

We describe our new dynamic approach to linear grouping for an item set $J' \subseteq J$ where *any* feasible solution can pack at most n' items of J'. We consider J' instead of J because some of our dynamic algorithms only use the dynamic linear grouping on a subset of items, e.g., big items.

Theorem 5.1. *Given a set J' with $|\textsc{Opt} \cap J'| \leq n'$ for all optimal solutions* Opt*, there is an algorithm with running time $\mathcal{O}\big(\frac{\log^5 n'}{\varepsilon^5}\big)$ that reduces the items in J' to item types $\mathcal{T}$ with $|\mathcal{T}| \leq \mathcal{O}\big(\frac{\log^2 n'}{\varepsilon^4}\big)$ and ensures $v(\textsc{Opt}_{\mathcal{T}}) \geq \frac{(1-\varepsilon)(1-2\varepsilon)}{(1+\varepsilon)^2} v(\textsc{Opt})$. Here, $\textsc{Opt}_{\mathcal{T}}$ is the optimal solution attainable by packing item types $\mathcal{T}$ instead of items in J' and using $J \setminus J'$ without any changes.*

Algorithm In the following, we use the notation X' for a set X to refer to $X \cap J'$ while X'' refers to $X \setminus J'$. Recall that, upon arrival, item values of items in J are rounded to natural powers of $(1+\varepsilon)$ to create the value classes V_ℓ where each item $j \in V_\ell$ is of value $(1+\varepsilon)^\ell$.

1) Let $\ell_{\max}$ be the guess for the highest value class with $V'_\ell \cap \textsc{Opt} \neq \emptyset$ and let $\bar{\ell} := \ell_{\max} - \left\lceil \frac{\log(n'/\varepsilon)}{\log(1+\varepsilon)} \right\rceil$.

2) For each value class $\bar{\ell} \leq \ell \leq \ell_{\max}$ and each guess $n_\ell = (1+\varepsilon)^l$ for $0 \leq l \leq \log_{1+\varepsilon} n'$ do the following: Consider the n_ℓ elements of V'_ℓ with the smallest size and determine the $\frac{1}{\varepsilon}$ many (almost) equal-sized groups $G_1(n_\ell), \ldots, G_{1/\varepsilon}(n_\ell)$ of $\lceil \varepsilon n_\ell \rceil$ or $\lfloor \varepsilon n_\ell \rfloor$ elements. If $\varepsilon n_\ell \notin \mathbb{N}$, ensure that $|G_k(n_\ell)| \leq |G_{k'}(n_\ell)| \leq |G_k(n_\ell)| + 1$ for $k \leq k'$. If $\frac{1}{\varepsilon}$ is not a natural power of $(1+\varepsilon)$, we also create the groups $G_1(\frac{1}{\varepsilon}), \ldots, G_{1/\varepsilon}(\frac{1}{\varepsilon})$ where $G_k(\frac{1}{\varepsilon})$ contains the kth smallest item in V'_ℓ.

   Let $G_1(n_\ell), \ldots, G_{1/\varepsilon}(n_\ell)$ be the corresponding groups sorted increasingly by the size of the items. Let $j_k(n_\ell) = \max\{j \in G_k(n_\ell)\}$ be the last index belonging to group $G_k(n_\ell)$. After having determined $j_k(n_\ell)$ for each possible value n_ℓ (including $\frac{1}{\varepsilon}$) and for each $1 \leq k \leq \frac{1}{\varepsilon}$, the size of each item j is rounded up to the size of the next large item j' where $j' = j_k(n_\ell)$ for some combination of k and n_ℓ.

3) Discard each item j with $j \in V'_\ell$ for $\ell < \bar{\ell}$.

Analysis Despite the new approach to apply linear grouping simultaneously to many possible values of n_ℓ, the analysis uses standard techniques. Thus, we only give a high-level overview of the proof here and refer the reader to Appendix C for the technical details. We start by observing that the loss in the objective function due to rounding item values to natural powers of $(1+\varepsilon)$ is bounded by a factor of $\frac{1}{1+\varepsilon}$ by Lemma 3.1. As $\bar{\ell}$ is chosen such that n' items of value at most $(1+\varepsilon)^{\bar{\ell}}$ contribute less than an ε-fraction of $\textsc{Opt}'$, the loss in the objective function by discarding items in value classes V'_ℓ with $\ell < \bar{\ell}$ is bounded by a factor $(1-\varepsilon)$. By taking only $(1+\varepsilon)^{\lfloor \log_{1+\varepsilon} n_\ell \rfloor}$ items of V'_ℓ instead of n_ℓ, we lose at most a factor of $\frac{1}{1+\varepsilon}$. Observing that the groups created by dynamic linear grouping are an actual refinement of the groups created by the classical linear grouping for a fixed number of items, we pack our items as done in linear grouping: Not packing the group with the largest items allows us to "shift" all rounded items of group $G_k(n_\ell)$ to the positions of the (not rounded) items in group $G_{k+1}(n_\ell)$ at the expense of losing a factor of $(1-2\varepsilon)$. Combining these results then shows the following lemma.

Lemma 5.2. *Let* $\textsc{Opt}$ *and* $\textsc{Opt}_{\mathcal{T}}$ *be as defined above. Then,* $v(\textsc{Opt}_{\mathcal{T}}) \geq \frac{(1-\varepsilon)(1-2\varepsilon)}{(1+\varepsilon)^2} v(\textsc{Opt})$.

As $\mathcal{T}$ contains at most $\frac{1}{\varepsilon}\left(\left\lceil \frac{\log n'/\varepsilon}{\log(1+\varepsilon)} \right\rceil + 1\right)$ many different value classes and using $\left\lceil \frac{\log n'}{\log(1+\varepsilon)} \right\rceil + 1$ many different values for $n_\ell = |\textsc{Opt} \cap V'_\ell|$ suffices as explained above, the next lemma follows.

Lemma 5.3. *The algorithm reduces the number of item types to* $\mathcal{O}(\frac{\log^2 n'}{\varepsilon^4})$.

The crucial ingredient for the refined dynamic linear grouping are the carefully calculated indices that indicate the largest item in a group for a value n_ℓ. These indices can be computed exactly once, depending on the current value n'. Combining this observation with the bound in Lemma 5.3 and the access times given in Lemma 3.2 is the main part of the proof of the next lemma.

Lemma 5.4. *For a given guess* $\ell_{\max}$*, the set* $\mathcal{T}^{(\ell_{\max})}$ *can be determined in time* $\mathcal{O}(\frac{\log^4 n'}{\varepsilon^4})$.

Proof of Theorem 5.1. Lemmas 3.1 and 5.2 bound the approximation ratio, Lemma 5.3 bounds the number of item types, and Lemma 5.4 bounds the running time of the dynamic linear grouping approach. □

5.2 A Dynamic Algorithm for Many Identical Knapsacks

In this section, we give a dynamic algorithm that achieves an approximation guarantee of $(1-\varepsilon)$ for MULTIPLE KNAPSACK with identical knapsack sizes. The running time of the update operation is always polynomial in $\log n$ and $\frac{1}{\varepsilon}$. In this section, we assume $m < n$ as otherwise assigning the items in some consistent order to the knapsacks is optimal. We focus on instances where m is large, i.e., $m \geq \frac{16}{\varepsilon^7}\log^2 n$. For $m \leq \frac{16}{\varepsilon^7}\log^2 n$, we use the algorithm for few knapsacks presented in Appendix B.

Theorem 5.5. *Let $U = \max\{Sm, nv_{\max}\}$. If $m \geq \frac{16}{\varepsilon^7}\log^2 n$, there is a dynamic algorithm for the* MULTIPLE KNAPSACK *problem with m identical knapsacks with approximation factor $(1-\varepsilon)$ and update time $\left(\frac{\log U}{\varepsilon}\right)^{\mathcal{O}(1)}$. Queries for single items and the solution value can be answered in time $\mathcal{O}\left(\frac{\log n}{\varepsilon}\right)^{\mathcal{O}(1)}$ and $\mathcal{O}(1)$, respectively. The whole solution P can be returned in time $|P|\left(\frac{\log n}{\varepsilon}\right)^{\mathcal{O}(1)}$.*

Definitions and Data Structures We partition the items into two sets, J_B, the *big* items, and J_S, the *small* items, with sizes $s_j \geq \varepsilon S$ and $s_j < \varepsilon S$, respectively. For an optimal solution OPT, define $\text{OPT}_B := \text{OPT} \cap J_B$ and $\text{OPT}_S := \text{OPT} \cap J_S$.

For this algorithm, we maintain three types of data structures: we store all items in one balanced binary tree in order of their arrivals, i.e., their indices. In this tree, we store the size s_j and the value v_j of each item j and additonally store the value class ℓ_j for big items. Big items are also stored in one balanced binary tree per value class V_ℓ sorted by non-decreasing size while all small items are sorted by non-decreasing density and stored in one tree. Overall, we have at most $2 + \log_{1+\varepsilon} v_{\max}$ many data structures to maintain. Upon arrival of a new item, we insert it into the tree of all items and classify this item as big or small according to $s_j \geq \varepsilon S_i$ or $s_j < \varepsilon S_i$. If the item is small, we insert it into the tree of small items. Otherwise, we determine its value class $\ell = \lfloor \log_{1+\varepsilon} v_j \rfloor$ and insert it into the corresponding tree.

Algorithm

1) **Linear grouping of big items:** Guess $\ell_{\max}$, the index of the highest value class that belongs to OPT_B and use dynamic linear grouping with $J' = J_B$ and $n' = \min\{\frac{m}{\varepsilon}, n_B\}$ to obtain $\mathcal{T}$, the set of item types t with their multiplicities n_t.
2) **Configurations:** Create all possible configurations of at most $\frac{1}{\varepsilon}$ big items and store these configurations in $\mathcal{C}$. For $c \in \mathcal{C}$ let v_c and s_c denote the total value respectively size of the items in c.
3) **Guessing the size of small items:** First guess v_S as estimate of $v(\text{OPT}_S)$. Let P be the maximal prefix of small items (sorted by decreasing density) with $v(P) < v_S$. Define $s_S := s(P)$.
4) **Configuration ILP:** Solve the following configuration ILP with variables y_c for $c \in \mathcal{C}$ for the current guesses $\ell_{\max}$ and s_S. Here, y_c counts how often a certain configuration c is used and n_{tc} denotes the number of items of type t in configuration c.

$$\begin{array}{llcll} \max & \sum_{c\in\mathcal{C}} y_c v_c & & & \\ \text{subject to} & \sum_{c\in\mathcal{C}} y_c s_c & \leq & (1-3\varepsilon)Sm - s_S & \\ & \sum_{c\in\mathcal{C}} y_c & \leq & (1-3\varepsilon)m & \\ & \sum_{c\in\mathcal{C}} y_c n_{tc} & \leq & n_t & \text{for all } t \in \mathcal{T} \\ & y_c & \in & \mathbb{Z}_{\geq 0} & \text{for all } c \in \mathcal{C} \end{array} \tag{P}$$

The first inequality ensures that the configurations chosen by the ILP fit into $(1-3\varepsilon)m$ knapsacks while reserving sufficient space for the small items. The second constraint limits the total number of configurations that are packed. The third inequality ensures that only available items are used.

5) **Obtaining an integral solution:** As m is large, we cannot solve the ILP optimally but need to relax the integrality constraint and allow fractional solutions. Given such a fractional solution, we round this fractional solution into an integral packing P_B using at most εm additional knapsacks while ensuring that $v(P_B) \geq v_{LP}$, where v_{LP} is the optimal solution value for the LP relaxation.

6) **Packing small items:** Consider the maximal prefix P of small items with $v(P) < v_S$ and let j be the densest small item not in P. Pack j into one of the knapsacks kept empty by P_B. Then, fractionally fill up the $(1-2\varepsilon)m$ knapsacks used by P_B and place any "cut" item into the εm additional knapsacks that are still empty. We can guarantee that this packing is feasible.

Analysis The first step is to analyze the loss in the objective function value due to the linear grouping. To this end, set $J' = J_B$ and $n' = \min\{\frac{m}{\varepsilon}, n\}$. Moreover, let $\textsc{Opt}_{\mathcal{T}}$ be the optimal packing when using the corresponding item types $\mathcal{T}$ obtained from applying dynamic linear rounding instead of the items in J_B. Then, the next corollary immediately follows from Theorem 5.1.

Corollary 5.6. *Let* $\textsc{Opt}$ *and* $\textsc{Opt}_{\mathcal{T}}$ *be defined as above. Then,* $v(\textsc{Opt}_{\mathcal{T}}) \geq \frac{(1-\varepsilon)(1-2\varepsilon)}{(1+\varepsilon)^2} v(\textsc{Opt})$.

In the next lemma, we show that there is a guess v_S with the corresponding size s_S such that $v^*_{\text{ILP}} + v_S + v_j$ for the optimal solution value v^*_{ILP} of (P) is a good guess for the optimal solution $v(\textsc{Opt}_{\mathcal{T}})$. Here, j is the densest small item not contained in P while P is the maximal prefix of small items with $v(P) < v_S$. The high-level idea of the proof is to restrict an optimal solution $\textsc{Opt}_{\mathcal{T}}$ to the $(1-3\varepsilon)m$ most valuable knapsacks and show that s_S underestimates the size of small items in these $(1-3\varepsilon)m$ knapsacks. Transforming these knapsacks into configurations then comprises a feasible solution for the configuration ILP. See Appendix D.

Lemma 5.7. *Let* v^*_{ILP} *and* $\textsc{Opt}_{\mathcal{T}}$ *be defined as above. There are* v_S *and* s_S *with* $v^*_{ILP} + v_S \geq \frac{1-3\varepsilon}{1+\varepsilon} v(\textsc{Opt}_{\mathcal{T}})$. *Moreover, for* P *and* j *as defined above,* $v(P) + v_j \geq v_S$.

Next, we explain how to approximately solve the LP relaxation of the configuration ILP (P) and round the solution to an integral packing in slightly more knapsacks. As any basic feasible solution of (P) has at most $\mathcal{O}\left(|\mathcal{T}|\right)$ strictly positive variables, solving its dual problem with the Grötschel-Lovasz-Schrijver [28] variant of the Ellipsoid method determines the relevant variables.

The separation problem is a KNAPSACK problem, which we only solve approximately in time polynomial in $\log n$ and $\frac{1}{\varepsilon}$. This approximate separation oracle correctly detects infeasibility while a solution that is declared feasible may only be feasible for a closely related problem causing a multiplicative loss in the objective function value of at most $(1-\varepsilon)$. We cannot use the approaches by Plotkin, Shmoys, and Tardos [67] and Karmarkar and Karp [48] directly as our configuration ILP contains two extra constraints which correspond to additional variables in the dual and, thus, to two extra terms in the objective function. Instead, we add an objective function constraint to the dual and test for feasibility for a set of geometrically increasing guesses of the objective function value. Given the maximal guess for which the dual is infeasible, we use the variables corresponding to constraints added by the Ellipsoid method to solve the primal. The multiplicative gap between the maximal infeasible and the minimal feasible such guess allows us to obtain a fractional solution with objective function value at least $\frac{1-\varepsilon}{1+\varepsilon}$. See Appendix D for the technical details.

Lemma 5.8. *Let* $U = \max\{Sm, nv_{\max}\}$. *Then, there is an algorithm that finds a feasible solution for the LP relaxation of* (P) *with value at least* $\frac{1-\varepsilon}{1+\varepsilon} v_{LP}$ *with running time bounded by* $\left(\frac{\log U}{\varepsilon}\right)^{\mathcal{O}(1)}$.

Having found a feasible solution with the Ellipsoid method, we use Gaussian elimination to obtain a basic feasible solution with no worse objective function value. By rounding down each variable in this solution, we obtain a feasible integral solution to the configuration ILP. For each configuration subjected to rounding, we place one additional configuration into one knapsack. As basic feasible solutions have at most $\mathcal{O}(|\mathcal{T}|)$ non-vanishing variables, the assumptions $\frac{16}{\varepsilon^7}\log^2 n \le m$ and $m < n$ imply $\frac{16}{\varepsilon^7}\log^2 m \le m$ which in turn guarantees $\mathcal{O}(|\mathcal{T}|) \le \varepsilon m$. Hence, the rounded solution uses at most $(1-2\varepsilon)m$ knapsacks and achieves a value of at least v_{LP}; see Appendix C.1.

Lemma 5.9. *If $\frac{16}{\varepsilon^7}\log^2 n \le m$, any feasible solution of the LP relaxation of (P) can be rounded to an integral solution using at most $(1-2\varepsilon)m$ knapsacks with total value at least v_{LP}.*

Next, we bound the value achieved by our algorithm in terms of the optimal solution.

Lemma 5.10. *Let P_F be the solution returned by our algorithm and let* Opt *be a current optimal solution. It holds that $v(P_F) \ge \frac{(1-\varepsilon)^2(1-2\varepsilon)(1-3\varepsilon)}{(1+\varepsilon)^4} v(\text{Opt})$.*

Proof. The solution found by our algorithm achieves the maximal value over all combinations of v_S, guesses of the value contributed by small items, and $\ell_{\max}$, the highest value class of a big item. Thus, it suffices to find a combination of v_S and $\ell_{\max}$ such that P, the corresponding packing, is feasible and satisfies $v(P) \ge \frac{(1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon)}{(1+\varepsilon)^4} v(\text{Opt})$.

Let Opt_B be the set of big items in Opt and let $\ell_{\max} := \max\{\ell : V_\ell \cap \text{Opt}_B \ne \emptyset\}$. For this guess $\ell_{\max}$, let $P_S \cup \{j\}$ be the set of small items of Lemma 5.7 such that $v^*_{\text{ILP}} + v(P_S) + v_j \ge \frac{1-3\varepsilon}{1+\varepsilon} v(\text{Opt}_{\mathcal{T}})$. By Lemma 5.9, there is a set of big items P_B with a feasible packing into $(1-2\varepsilon)m$ knapsacks with total value at least $\frac{1-\varepsilon}{1+\varepsilon} v_{\text{ILP}}$. Packing j on its own and P_S in a First Fit manner, we extend this to a feasible packing of $P_B \cup P_S \cup \{j\}$ into $(1+\varepsilon)(1-2\varepsilon)m + 1 \le m$ knapsacks; see Lemma C.5. With Lemma 5.7,

$$v(P_F) \ge v(P) \ge \frac{1-\varepsilon}{1+\varepsilon} v_{\text{ILP}} + v_S + v_j \ge \frac{(1-\varepsilon)(1-3\varepsilon)}{(1+\varepsilon)^2} v(\text{Opt}_{\mathcal{T}}),$$

where $\text{Opt}_{\mathcal{T}}$ is the most valuable packing after linear grouping. With Corollary 5.6 we get

$$v(P_F) \ge \frac{(1-\varepsilon)^2(1-2\varepsilon)(1-3\varepsilon)}{(1+\varepsilon)^4} v(\text{Opt}). \tag{1}$$

□

The next lemma bounds the running time of our algorithm. The proof follows from the fact that the algorithm considers at most $\mathcal{O}\left(\frac{\log(nv_{\max})\log v_{\max}}{\varepsilon^2}\right)$ many rounds, the running time for dynamic linear grouping bounded in Lemma 5.4, and the running time for solving the configuration ILP as described in Lemma 5.8.

Lemma 5.11. *Let $U := \max\{Sm, nv_{\max}\}$. The running time of our algorithm is bounded by $\left(\frac{\log U}{\varepsilon}\right)^{\mathcal{O}(1)}$.*

Queries We show how to efficiently answer different queries and give their runtime. In contrast to the previous section, for transforming an implicit solution into an explicit packing it does not suffice to know whether an item is packed or not but the query operation has to compute the knapsack where item j is packed. Since we do not explicitly store the packing of any item, we define and update pointers for small items and for each item type that dictate the knapsacks where the corresponding items are packed. Therefore, to stay consistent for the precise packing of a particular item between two update operations, we additionally cache query answers for the current round in the data structure that store items.

- **Single Item Query:** If the queried item is small, we check if it belongs to the prefix of densest items that is part of our solution. In this case, the pointer for small items determines the knapsack. If the queried item is big, we retrieve its item type and check if it belongs to smallest items of this type that are packed by the implicit solution. In this case, the pointer for this item type dictates the knapsack.
- **Solution Value Query:** As the algorithm works with rounded values, after having found the current solution, we use prefix computation on the small items and on any value class of big items, to calculate and store the actual solution value. On query, we return the stored solution value.
- **Query Entire Solution:** We use prefix computation on the small items as well as on the value classes of the big items to determine the packed items. Then, we use the Single Item Query to determine their respective knapsacks.

For a detailed analysis of the queries and the proof of the following lemma, see Appendix D.

Lemma 5.12. *The solution determined by the query algorithms is feasible and achieves the claimed total value. The query times of our algorithm are as follows.*

(i) Single item queries can be answered in time $\mathcal{O}\big(\max\big\{\log\frac{\log n}{\varepsilon}, \frac{1}{\varepsilon}\big\}\big)$
(ii) Solution value queries can be answered in time $\mathcal{O}(1)$
(iii) Queries of the entire solution P *are answered in time* $\mathcal{O}\big(|P|\max\big\{\log\frac{\log n}{\varepsilon}, \frac{1}{\varepsilon}\big\}\big)$.

Proof of Theorem 5.5. In Lemma 5.10, we calculate the approximation ratio achieved by our algorithm. For calculating the update time and the time needed for answering queries, observe that $|\mathcal{C}'|$ depends on m. Then, Lemma 5.11 gives the desired bounds on the update time while Lemma 5.12 bounds the time needed for answering a query. Lemma 5.12 also guarantees that the query answers are correct. ☐

6 Solving MULTIPLE KNAPSACK

Having laid the groundwork with the previous two sections, we finally show how to maintain solutions for arbitrary instances of the MULTIPLE KNAPSACK problem, and give the main result of this paper. A detailed proof can be found in Appendix F.

Theorem 6.1. *For* $\varepsilon > 0$*, there exists a dynamic algorithm for* MULTIPLE KNAPSACK *that maintains an* $(1-\varepsilon)$*-approximate solution in update time* $\big(\frac{1}{\varepsilon}\log(nv_{\max})\big)^{f(1/\varepsilon)} + \mathcal{O}(\frac{1}{\varepsilon}\log\overline{v}\log n)$*, with* f *quasi-linear. Item queries are served in time* $\mathcal{O}(\frac{\log n}{\varepsilon^2})$ *and the whole solution* P *can be output in time* $\mathcal{O}(\frac{\log^4 n}{\varepsilon^6}|P|)$.

As mentioned in Section 2, we obtain this result by partitioning the knapsacks into three sets, special, extra and ordinary knapsacks, and solving the respective subproblems. This has similarities to the approach in [43]; however, there it was sufficient to have only two groups of knapsacks. In Appendices B and E we show the following two theorems. They solve the special and ordinary subproblems by applying the algorithmic techniques developed in Sections 4 and 5 respectively.

Theorem 6.2. *For* $\varepsilon > 0$*, there is a dynamic algorithm for* MULTIPLE KNAPSACK *with* m *knapsacks that achieves an approximation factor of* $(1-\varepsilon)$ *with update time* $2^{f(1/\varepsilon)}(\frac{m}{\varepsilon}\log(nv_{\max}))^{\mathcal{O}(1)} + \mathcal{O}(\frac{1}{\varepsilon}\log\overline{v}\log n)$*, with* f *quasi-linear. Moreover, item queries are answered in time* $\mathcal{O}(\log\frac{m^2}{\varepsilon^6})$*, solution value queries in time* $\mathcal{O}(1)$ *and queries of a single knapsack or the entire solution in time linear in the size of the output.*

Theorem 6.3. *There is a dynamic algorithm for* MULTIPLE KNAPSACK *that, when given* $L = (\frac{\log n}{\varepsilon})^{O(1/\varepsilon)}$ *additional knapsacks as resource augmentation, achieves an approximation factor* $(1-\varepsilon)$ *with update time* $(\log n)^{\mathcal{O}(1/\varepsilon)}(\log S_{\max}\log v_{\max})^{\mathcal{O}(1)}$ *where* $S_{\max} := \max\{S_i : i \in [m]\}$*. Item queries are answered in time* $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$ *and a solution* P *can be output in time* $\mathcal{O}\big(|P|\frac{\log^4 n}{\varepsilon^6}\big)$.

Definitions and Data Structures. We assume that $m > (\frac{1}{\varepsilon})^{4/\varepsilon} \cdot L$. Otherwise, we apply Theorem 6.3. Consider $\frac{1}{\varepsilon}$ groups of knapsacks with sizes $\frac{L}{\varepsilon^{3i}}$ for $i \in \{0, 1, \ldots, \frac{1}{\varepsilon} - 1\}$ such that the first group ($i = 0$) consists of the L largest knapsacks, the second ($i = 1$) of the $\frac{L}{\varepsilon^3}$ next largest, and so on. In OPT, one of these groups contains items with total value at most $\varepsilon \cdot \text{OPT}$. Let $k \in \{0, 1, \ldots, \frac{1}{\varepsilon} - 1\}$ be the index of such a group and let $L_S := \sum_{i=0}^{k-1} \frac{L}{\varepsilon^{3i}}$. Based on k and L_S, we partition the knapsacks into three groups that increase in knapsack capacity but decrease in cardinality. We refer to them as *special*, *extra* and *ordinary* knapsacks, with the special knapsacks being the L_S largest ones. The extra knapsacks are the $\frac{L}{\varepsilon^{3k}} > \frac{L_S}{\varepsilon^2} + L$ next largest, and the ordinary knapsacks the remaining ones.

Call an item *ordinary* if it fits into the largest ordinary knapsack and *special* otherwise. Denote by J_O and J_S the set of ordinary and special items respectively and by S_O the total size, rounded down to a power of $(1 + \varepsilon)$, of ordinary items that OPT places in special knapsacks.

Since we use the algorithms from Appendices B and E as subroutines, we require the maintenance of the corresponding data structures. This causes the additive term $\mathcal{O}(\frac{1}{\varepsilon} \log \overline{v} \log n)$ in the update time.

Algorithm.

1) **Dynamic linear grouping:** Compute $\mathcal{O}(\frac{\log^2 n}{\varepsilon^4})$ item types as in Section 5.1 (using $J' = J$ and $n' = n$). Guess k and determine whether items of a certain type are ordinary or special.

2) **High-value ordinary items:** Place each of the $\frac{L_S}{\varepsilon^2}$ most valuable ordinary items in an empty extra knapsack. On a tie choose the larger item. Denote the set of such items by J_E.

3) **Virtual ordinary knapsack:** Guess S_O and add a virtual knapsack with capacity S_O to the ordinary subproblem. In the LP (P) of Appendix E, that is used to solve the ordinary subproblem, only consider variables $z_{g,t}$ for this knapsack but no configurations.

4) **Solve ordinary instance:** Remove until the next update, from data structures used in the ordinary subproblem, the set J_E of items placed in extra knapsacks in Step 2). Solve the subproblem with the virtual knapsack as in Appendix E and use extra knapsacks for resource augmentation. When rounding up variables, fill the $\mathcal{O}(\frac{\log^4 n}{\varepsilon^8})$ rounded items from the virtual knapsack into extra knapsacks.

5) **Create bundles** Consider the items that remain on the virtual ordinary knapsack after rounding. Sort them by type (e.g., first value, then size) and cut them fractionally to form $\frac{L_S}{\varepsilon}$ bundles of equal size. Denote by B_O this set of bundles and, for each bundle, remember how many items of each type are placed entirely inside it. Place fractionally cut items into extra knapsacks. Consider each $B \in B_O$ as an item of size and value equal to the fractional size respectively value of items placed entirely in B.

6) **Solve special instance:** Temporarily insert the bundles in B_O into the data structures used in the special subproblem. Solve this subproblem as detailed in Appendix B.

7) **Implicit solution:** Among all guesses, keep the solution P_F with the highest value. Store items in J_E and their placement explicitly. Revert the removal of J_E from the ordinary data structures only during the next update. For the remaining items, the solutions are given as in the respective subproblem, see Appendices B and E with the exception of items packed in the virtual ordinary knapsack. The solution on these items is stored implicitly by deciding membership in a bundle on a query.

Queries For handling queries, we essentially use the same approach as in Appendices B and E for the ordinary and special subproblem respectively. However, special care has to be taken with items in the virtual knapsack. In the ordinary subproblem, we assume that items of a certain type which are packed in the virtual knapsack are the first, i.e., smallest, of that type. We can therefore decide in constant time whether or not an item is contained in the virtual knapsack and, if this is the case, fill it into the free space in special knapsacks reserved by bundles. We do this efficiently by using a first fit algorithm on the knapsacks with reserved space. Since items on extra knapsacks are stored explicitly, they can be accessed in constant time. Note that the number of special knapsacks is in $(\frac{\log n}{\varepsilon})^{O(1/\varepsilon)}$, so the function g in Appendix B is in $\mathcal{O}\left(\frac{1}{\varepsilon}\right)$.

Lemma 6.4. *The query times of our algorithm are as follows.*
(i) Single item queries are answered in time $\mathcal{O}(\frac{\log n}{\varepsilon^2})$.
(ii) Solution value queries are answered in time $\mathcal{O}(1)$.
(iii) Queries of the entire solution P *are answered in time* $\mathcal{O}(\frac{\log^4 n}{\varepsilon^6}|P|)$.

7 Conclusion

We have presented a robust dynamic framework for KNAPSACK and MULTIPLE KNAPSACK with item and knapsack arrivals and departures and queries on solution size and item presence in the solution. By having n items arrive one by one, any dynamic algorithm can be turned into a non-dynamic framework with incurring an additional linear term in the running time. Hence, the performance of any dynamic framework is subject to the same lower bounds as non-dynamic approximation schemes. Our results on KNAPSACK and MULTIPLE KNAPSACK with identical capacities are tight in the sense that their runtime is a form of FPTAS (resp. EPTAS), matching the known results for the approximation algorithms.

Clearly, further generalizing the results beyond MULTIPLE KNAPSACK remains an interesting open question. A straightforward generalization is d-dimensional KNAPSACK where each item comes with d different sizes corresponding to d dimensions, each knapsack has d different capacities, and a feasible packing of a subset of items must meet the capacity constraint in each dimension. Using the reduction to the 1-dimensional setting developed by [22], allows us to generalize our results for KNAPSACK and MULTIPLE KNAPSACK with identical knapsacks to the d-dimensional setting at the loss of a factor d in the approximation guarantee. A result by [55] shows that, unless W[1] $=$ FPT, 2-dimensional knapsack does not admit a dynamic algorithm maintaining a $(1-\varepsilon)$-approximation in worst-case update time $f\left(\frac{1}{\varepsilon}\right)n^{\mathcal{O}(1)}$. Designing a dynamic framework with a better guarantee than $\frac{1}{d}$ remains open.

We hope to foster further research within the dynamic algorithm framework for other packing, scheduling and, generally, non-graph problems. For bin packing and for scheduling to minimize the makespan on uniformly related machines, we notice that existing PTAS techniques from [48] and [36, 44] combined with rather straightforward data structures can be lifted to a fully dynamic framework for the respective problems.

Appendices

A Proofs for Single Knapsack

In this section we give the detailed analysis of our algorithm for KNAPSACK in Section 4. We consider the iteration in which the guesses $V_{\ell_{\max}}, V_{\ell_{\min}}, n_{\min}$ and $\mathcal{V}_L$ are correct and show that the obtained solution has a value of at least $(1-4\varepsilon)\cdot v(\textsc{Opt})$.

Let $\mathcal{P}_1$ be the set of solutions respecting: (i) packed items not in $H_{\frac{1}{\varepsilon}}$ have a value of at most $V_{\ell_{\min}}$ but are not part of the $n_{\min}$ smallest items of the value class $V_{\ell_{\min}}$, and (ii) the total value of these items lies in $[\mathcal{V}_L, (1+\varepsilon)\mathcal{V}_L]$. Denote by $\textsc{Opt}_1$ the solution of highest value in $\mathcal{P}_1$.

Lemma A.1. *Consider* $\textsc{Opt}_1$ *defined as above. Then,* $v(\textsc{Opt}_1) \geq (1-\varepsilon)\cdot v(\textsc{Opt})$.

Proof. Let $\textsc{Opt}^*$ be the packing obtained from $\textsc{Opt}$ by removing all items belonging to $\textsc{Opt}_{\frac{1}{\varepsilon}}$ whose value is strictly smaller than $\varepsilon^2 V_{\ell_{\max}}$. Since $\textsc{Opt}_{\frac{1}{\varepsilon}}$ consists of $\frac{1}{\varepsilon}$ many items, the total value of removed items is at most $\frac{1}{\varepsilon}\cdot\varepsilon^2 V_{\ell_{\max}} \leq \varepsilon\cdot\textsc{Opt}$. We show that $\textsc{Opt}^* \in \mathcal{P}_1$.

Consider an item j in $\textsc{Opt}_{\frac{1}{\varepsilon}}$ of value $v_j \geq \varepsilon^2 V_{\ell_{\max}}$. If $v_j = V_{\ell_{\min}}$, then $j \in H_{\frac{1}{\varepsilon}}$ by definition of $n_{\min}$ and $\textsc{Opt}_{\frac{1}{\varepsilon}}$, specifically, due to the tie-breaking rules. Assume now that $v_j > V_{\ell_{\min}}$ and $j \notin H_{\frac{1}{\varepsilon}}$. Recall that $H_{\frac{1}{\varepsilon}}$ contains the $\frac{1}{\varepsilon}$ smallest items of value v_j, and $|\textsc{Opt}_{\frac{1}{\varepsilon}}| = \frac{1}{\varepsilon}$. Thus, there exists an item of value v_j, smaller than j, which belongs to $H_{\frac{1}{\varepsilon}}$ but not to $\textsc{Opt}_{\frac{1}{\varepsilon}}$. Exchanging j for this item contradicts the definition of $\textsc{Opt}$. Therefore, $j \in H_{\frac{1}{\varepsilon}}$ and Condition (i) is satisfied. Condition (ii) follows directly from the definition of $\mathcal{V}_L$, and therefore $\textsc{Opt}^* \in \mathcal{P}_1$, concluding the proof. □

Lemma A.2. *Let* $\textsc{Opt}_2$ *be the optimal solution of the instance* I *on which the FPTAS is run at Step 3). Then,* $v(\textsc{Opt}_2) \geq (1-\varepsilon)\cdot v(\textsc{Opt}_1)$.

Proof. Consider the fractional solution $\textsc{Opt}_1^*$ for I that is obtained from $\textsc{Opt}_1$ as follows. Place items from $H_{\frac{1}{\varepsilon}}$ as in $\textsc{Opt}_1$ and additionally place the placeholder item B. Denote by J_L the set of items packed by $\textsc{Opt}_1$ that are not in $H_{\frac{1}{\varepsilon}}$, i.e., the low-value items. By definition of B, we have $v_B = \mathcal{V}_L \geq (1-\varepsilon)v(J_L)$. Further, since B consists of the densest low-value items, it must be the case, that $s_B \leq s(J_L)$. Therefore, $\textsc{Opt}_1^*$ is a feasible solution for I and the statement follows. □

Lemma A.3. *For the solution* P_F *of the algorithm, we have* $v(P_F) \geq (1-4\varepsilon)\cdot v(\textsc{Opt})$.

Proof. The solution P_{FPTAS} returned by the FPTAS in Step 3) has a value of at least $(1-\varepsilon)\cdot v(\textsc{Opt}_2)$. The solution P_F is obtained from P_{FPTAS} by replacing the placeholder with the corresponding low-value items, except possibly the fractional item j. Since there are $\frac{1}{\varepsilon}$ items in $\textsc{Opt}$ that are of higher value than j, namely the ones in $\textsc{Opt}_{\frac{1}{\varepsilon}}$, this implies

$$v(P_F) \geq v(P_{\text{FPTAS}}) - \varepsilon\cdot v(\textsc{Opt}).$$

Using Lemmas A.1 and A.2, we obtain:

$$\begin{aligned} v(P_F) &\geq v(P_{\text{FPTAS}}) - \varepsilon\cdot v(\textsc{Opt}) \\ &\geq (1-\varepsilon)^2\cdot v(\textsc{Opt}_1) - \varepsilon\cdot v(\textsc{Opt}) \\ &\geq (1-\varepsilon)^3\cdot v(\textsc{Opt}) - \varepsilon\cdot v(\textsc{Opt}) \\ &\geq (1-4\varepsilon)\cdot v(\textsc{Opt}). \end{aligned}$$

□

Lemma A.4. *The algorithm has update time* $\mathcal{O}\big(\frac{1}{\varepsilon^9}\cdot\log n\cdot\log(n\cdot v_{\max})\cdot\log^2 v_{\max}\big)$.

Proof. In the first step, guessing $V_{\ell_{\max}}$ and $V_{\ell_{\min}}$, and therefore enumerating over all possible values, leads to $\mathcal{O}(\frac{1}{\varepsilon^2} \cdot \log^2 v_{\max})$ many iterations. Guessing $n_{\min}$ adds an additional factor of $\frac{1}{\varepsilon}$.

In the second step, again guessing $\mathcal{V}_L$ adds a factor to the runtime, specifically $\mathcal{O}(\frac{1}{\varepsilon}\log(n \cdot v_{\max}))$. Temporarily removing the $n_{\min} \leq \frac{1}{\varepsilon}$ elements from the data structure costs a total of $\mathcal{O}(\frac{1}{\varepsilon}\log n)$, as does adding back removed items from a previous iteration. Computing the size of B can be done by querying the prefix of value just above $\mathcal{V}_L$ in time $\mathcal{O}(\log n)$, see Section 3.

For Step 3) note that the set $H_{\frac{1}{\varepsilon}}$ spans value classes ranging from $\varepsilon^2 \cdot V_{\ell_{\max}}$ or higher to $V_{\ell_{\max}}$. As values are rounded to powers of $(1+\varepsilon)$, we consider at most $\log_{1+\varepsilon}\frac{1}{\varepsilon^2}$ many. Hence, $H_{\frac{1}{\varepsilon}}$ is composed of $\mathcal{O}(\frac{1}{\varepsilon^3})$ items and the FPTAS runs in time $\mathcal{O}\big(\big(\frac{1}{\varepsilon^{9/4}}\frac{1}{\varepsilon^{3/2}} + \frac{1}{\varepsilon^2}\big)/2^{\Omega(\sqrt{\log(1/\varepsilon)})}\big) = \mathcal{O}(\frac{1}{\varepsilon^4})$.

Recall, that we need to maintain one data structure for every existing and one for each possible value class, that is, $\mathcal{O}(\frac{1}{\varepsilon}\log\overline{v})$ many data structures in total. Maintenance of these, i.e., insertion or deletion of an item, takes time $\mathcal{O}(\frac{1}{\varepsilon}\log\overline{v}\log n)$ in total. □

B Few Different Knapsacks

It is not very difficult to extend the approach from Section 4 to the case of multiple but few knapsacks. While theoretically applicable for any number of knapsacks, the runtime is reasonable when $m = (\frac{1}{\varepsilon}\log n)^{\mathcal{O}_\varepsilon(1)}$. The main difference to Section 4 comes from the fact that to reserve space for low-value items, a single placeholder is no longer sufficient. Instead, we utilize several smaller placeholders. And since guessing the size of low-value items for every knapsack would lead to a runtime exponential in m, we instead employ a sufficiently large number of placeholder items, namely $\frac{m}{\varepsilon}$ many.

This leads to additional changes as there are more fractionally cut items, i.e., one per placeholder. To be able to charge them as before in Lemma A.3, we now consider the $\frac{m}{\varepsilon^2}$ most profitable items in OPT. This in turn leads to a larger candidate set of size $\frac{m}{\varepsilon^2}$. Furthermore, since we consider multiple knapsacks, we need to utilize an EPTAS instead of an FPTAS. Besides these changes, the algorithm remains unchanged.

Theorem 6.2. *For $\varepsilon > 0$, there is a dynamic algorithm for* MULTIPLE KNAPSACK *with m knapsacks that achieves an approximation factor of $(1-\varepsilon)$ with update time $2^{f(1/\varepsilon)}(\frac{m}{\varepsilon}\log(nv_{\max}))^{\mathcal{O}(1)} + \mathcal{O}(\frac{1}{\varepsilon}\log\overline{v}\log n)$, with f quasi-linear. Moreover, item queries are answered in time $\mathcal{O}(\log\frac{m^2}{\varepsilon^6})$, solution value queries in time $\mathcal{O}(1)$ and queries of a single knapsack or the entire solution in time linear in the size of the output.*

Definitions and Data Structures Let OPT be the set of items used in an optimal solution and $\text{OPT}_{\frac{m}{\varepsilon^2}}$ the set containing the $\frac{m}{\varepsilon^2}$ most valuable items of OPT; in both cases, break all ties by picking smaller-size items. Further, denote by $V_{\ell_{\max}}$ and $V_{\ell_{\min}}$ the highest and lowest value (class) of an element in $\text{OPT}_{\frac{m}{\varepsilon^2}}$ respectively and by $n_{\min}$ the number of elements of $\text{OPT}_{\frac{m}{\varepsilon^2}}$ with value $V_{\ell_{\min}}$. Let $\mathcal{V}_L$ be the total value of the items in $\text{OPT} \setminus \text{OPT}_{\frac{m}{\varepsilon^2}}$, rounded down to a power of $(1+\varepsilon)$. The data structures used are identical to those of Section 4.

Algorithm

1) **Compute high-value candidates $H_{\frac{m}{\varepsilon^2}}$:** Guess the three values $V_{\ell_{\max}}$, $V_{\ell_{\min}}$ and $n_{\min}$. If $V_{\ell_{\min}} \cdot m \geq \varepsilon^3 \cdot V_{\ell_{\max}}$, then define $H_{\frac{m}{\varepsilon^2}}$ to be the set that contains the $\frac{m}{\varepsilon^2}$ smallest items of each of the value classes $V_{\ell_{\min}+1}, \ldots, V_{\ell_{\max}}$, plus the $n_{\min}$ smallest items from $V_{\ell_{\min}}$.
Otherwise, we set $H_{\frac{m}{\varepsilon^2}}$ to be the union of the $\frac{m}{\varepsilon^2}$ smallest items of each of the value classes with values in $[\frac{\varepsilon^3}{m} \cdot V_{\ell_{\max}}, V_{\ell_{\max}}]$.

2) **Create bundles of low-value items as placeholders:** Guess the value $\mathcal{V}_L$ and consider the data structure containing all the items of value at most $V_{\ell_{\min}}$ sorted by decreasing density. Remove from

it (temporarily) the $n_{\min}$ smallest items of value $V_{\ell_{\min}}$. Insert them back into the data structure right before the next full update. From the remaining items, compute the size of fractional items necessary to reach a value of $\mathcal{V}_L$. That is, sum the sizes of the densest items until the total value exceeds $\mathcal{V}_L$ and, if necessary, cut the last item fractionally. Cut this range of items again fractionally to obtain bundles $B_1, B_2, \dots, B_{\frac{m}{\varepsilon}}$ of equal size $\frac{\varepsilon}{m} \cdot \mathcal{V}_L$. The value of a bundle is determined by the fractional value of the contained items.

3) **Use an EPTAS:** Consider the instance I consisting of the items in $H_{\frac{m}{\varepsilon^2}}$ and the placeholder bundles $B_1, B_2, \dots, B_{\frac{m}{\varepsilon}}$. Run the EPTAS designed by Jansen [43, 45], parameterized by ε, to obtain a packing P for this instance.

4) **Implicit Solution:** Among all guesses, keep the feasible solution P with the highest value. Then, for any knapsack, place into the knapsack items from $H_{\frac{m}{\varepsilon^2}}$ as in P and, if B_k is placed in P on this knapsack, also place the low-value items that constitute B_k, except possibly items cut fractionally. While used candidates can be stored explicitly, low-value items are given only implicitly by saving the correct guesses and recomputing B_k on a query.

Analysis. The analysis is almost identical with that of Section 4 with only slight changes to accommodate the alterations described above. For completeness, we give the full proofs. We consider the iteration in which all guesses $(V_{\ell_{\max}}, V_{\ell_{\min}}, n_{\min}, \mathcal{V}_L)$ are correct, and show that the obtained solution has a value of at least $(1 - 6\varepsilon) \cdot v(\text{OPT})$. To this end, we consider intermediate results to analyze the impact of each step.

Let $\mathcal{P}_1$ be the set of solutions respecting: (i) items not in $H_{\frac{m}{\varepsilon^2}}$ have a value of at most $V_{\ell_{\min}}$ but are not part of the $n_{\min}$ smallest items of the value class $V_{\ell_{\min}}$, and (ii) the total value of these items lies in $[\mathcal{V}_L; (1+\varepsilon)\mathcal{V}_L]$. Denote by OPT_1 the solution of highest value in $\mathcal{P}_1$.

Lemma B.1. *Consider* OPT_1 *defined as above. Then,* $v(\text{OPT}_1) \geq (1 - \varepsilon) \cdot v(\text{OPT})$.

Proof. Let OPT^* be the packing obtained from OPT by removing all items belonging to $\text{OPT}_{\frac{m}{\varepsilon^2}}$ whose value is strictly smaller than $\frac{\varepsilon^3}{m} \cdot V_{\max}$. Since $\text{OPT}_{\frac{m}{\varepsilon^2}}$ consists of $\frac{m}{\varepsilon^2}$ many items, the total value of removed items is at most $\frac{m}{\varepsilon^2} \cdot \frac{\varepsilon^3}{m} \cdot V_{\max} \leq \varepsilon \cdot \text{OPT}$. We show that $\text{OPT}^* \in \mathcal{P}_1$.

Consider an item j in $\text{OPT}_{\frac{m}{\varepsilon^2}}$ of value $v_j \geq \frac{\varepsilon^3}{m} \cdot V_{\max}$. If $v_j = V_{\ell_{\min}}$, then $j \in H_{\frac{m}{\varepsilon^2}}$ by definition of $n_{\min}$ and $\text{OPT}_{\frac{m}{\varepsilon^2}}$; specifically, due to the tie-breaking rules. Assume now that $v_j > V_{\ell_{\min}}$ and $j \notin H_{\frac{m}{\varepsilon^2}}$. Recall that $H_{\frac{m}{\varepsilon^2}}$ contains the $\frac{m}{\varepsilon^2}$ smallest items of value v_j, and $|\text{OPT}_{\frac{m}{\varepsilon^2}}| = \frac{m}{\varepsilon^2}$. Thus, there exists an item of value v_j, smaller than j, which belongs to $H_{\frac{m}{\varepsilon^2}}$ but not to $\text{OPT}_{\frac{m}{\varepsilon^2}}$. Exchanging j for this item contradicts the definition of OPT. Therefore, $j \in H_{\frac{m}{\varepsilon^2}}$ and Condition (i) is satisfied. Condition (ii) follows directly from the definition of V_L, and therefore $\text{OPT}^* \in \mathcal{P}_1$, concluding the proof. □

Lemma B.2. *Let* OPT_2 *be the optimal solution of the instance* I *on which the EPTAS is run at Step 3). Then,* $v(\text{OPT}_2) \geq (1 - 2\varepsilon) \cdot v(\text{OPT}_1)$.

Proof. Consider the fractional solution OPT_1^* for I that is obtained from OPT_1 as follows. First, place items from $H_{\frac{m}{\varepsilon^2}}$ as in OPT_1. Next, consider the placeholder bundles $B_1, B_2, \dots, B_{\frac{m}{\varepsilon}}$ in any order, and place them fractionally into the remaining space. That is, place remaining bundles in the first non-full knapsack. If a bundle does not fit, fill the current knapsack with a fraction of the bundle and place the remaining fraction in the next non-full knapsack using the same process. Finally, discard the fractionally cut bundles.

Denote by J_L the set of items packed by OPT_1 that are not in $H_{\frac{m}{\varepsilon^2}}$, i.e., the low-value items. Since the bundles consists of the densest low-value items, it must be the case, that $\sum_{k=1}^{m} s(B_k) \leq s(J_L)$. Therefore, OPT_1^* is a feasible solution for I and the statement follows.

By definition of the bundles, we have $\sum_{k=1}^{m} v(B_k) = V_L \geq (1 - \varepsilon) v(J_L)$. Further, since there are $\frac{m}{\varepsilon}$ bundles of equal value and at most m of them are cut fractionally and discarded, we conclude that $v(\text{OPT}_1^*) \geq (1 - 2\varepsilon) \cdot v(\text{OPT}_1)$. □

Lemma B.3. *For the solution P_F of the algorithm, we have $v(P_F) \geq (1-6\varepsilon) \cdot v(\textsc{Opt})$.*

Proof. The solution P_{EPTAS} returned by the EPTAS in Step 3) has a value of at least $(1-\varepsilon) \cdot v(\textsc{Opt}_2)$. The solution P_F is obtained from P_{EPTAS} by replacing the placeholder bundles with the corresponding low-value items with the exception of fractionally cut ones, of which there are at most $\frac{m}{\varepsilon}$ many. Since there are $\frac{m}{\varepsilon^2}$ items in $\textsc{Opt}$ that are of higher value than these items, namely the ones in $\textsc{Opt}_{\frac{m}{\varepsilon^2}}$, this implies

$$v(P_F) \geq v(P_{\text{EPTAS}}) - \varepsilon \cdot v(\textsc{Opt}).$$

Using Lemmas B.1 and B.2, we obtain:

$$\begin{aligned} v(P_F) &\geq v(P_{\text{EPTAS}}) - \varepsilon \cdot v(\textsc{Opt}) \\ &\geq (1-2\varepsilon)^2 \cdot v(\textsc{Opt}_1) - \varepsilon \cdot v(\textsc{Opt}) \\ &\geq (1-2\varepsilon)^2 \cdot (1-\varepsilon) \cdot v(\textsc{Opt}) - \varepsilon \cdot v(\textsc{Opt}) \\ &\geq (1-4\varepsilon) \cdot (1-\varepsilon) - \varepsilon) \cdot v(\textsc{Opt}) \\ &\geq (1-6\varepsilon) \cdot v(\textsc{Opt}), \end{aligned}$$

where the second to last equation follows from Bernoulli's inequality. □

Lemma B.4. *The algorithm has an update time of $2^{\mathcal{O}(\frac{1}{\varepsilon} \cdot \log^4(\frac{1}{\varepsilon}))} \cdot (\frac{m}{\varepsilon} \log(n v_{\max}))^{\mathcal{O}(1)} + \mathcal{O}(\frac{1}{\varepsilon} \log \overline{v} \log n)$.*

Proof. In the first step, guessing $V_{\ell_{\max}}$ and $V_{\ell_{\min}}$ leads to $\mathcal{O}(\frac{1}{\varepsilon^2} \log^2 v_{\max})$ many iterations. Guessing $n_{\min}$ adds an additional factor of $\frac{m}{\varepsilon^2}$. In the second step, guessing $\mathcal{V}_L$ leads to $\mathcal{O}(\frac{1}{\varepsilon} \log(n v_{\max}))$ many additional iterations, so the factor due to guessing is $\mathcal{O}(\frac{m}{\varepsilon^5} \log^2 v_{\max} \log(n v_{\max}))$

Temporarily removing the $n_{\min} \leq \frac{m}{\varepsilon^2}$ elements from the data structure costs a total of $\mathcal{O}(\frac{m}{\varepsilon^2} \log n)$, as does adding back removed items from a previous iteration. Computing the size of the bundles can be done by querying the prefixes of value just above $\mathcal{V}_L$, so in time $\mathcal{O}(\log n)$. Computing the cut items of the bundles takes time $\frac{m}{\varepsilon} \log n$.

The set $H_{\frac{m}{\varepsilon^2}}$ spans value classes ranging from $V_{\ell_{\max}}$ to a value at least $\frac{\varepsilon^3}{m} \cdot V_{\ell_{\max}}$. As the value classes correspond to powers of $(1+\varepsilon)$, this means we consider at most $\log_{1+\varepsilon} \frac{m}{\varepsilon^3}$ many. Since each of them contains at most $\frac{m}{\varepsilon^2}$ items, $H_{\frac{m}{\varepsilon^2}}$ contains $\mathcal{O}(\frac{m^2}{\varepsilon^6})$ items in total. Thus, in the third step, the EPTAS, used on $\mathcal{O}(\frac{m^2}{\varepsilon^6})$ many items, runs in time $2^{\mathcal{O}(\frac{1}{\varepsilon} \cdot \log^4(\frac{1}{\varepsilon}))} + (\frac{m}{\varepsilon})^{\mathcal{O}(1)}$. Together, this gives the desired update time.

Recall, that we need to maintain one data structure for every existing and one for each possible value class, that is, $\mathcal{O}(\frac{1}{\varepsilon} \log \overline{v})$ many data structures in total. Maintaining these takes time $\mathcal{O}(\frac{1}{\varepsilon} \log \overline{v} \log n)$. □

Queries We show how to efficiently handle the different types of queries and state their runtime.

- **Single Item Query:** If the queried item is contained in $H_{\frac{m}{\varepsilon^2}}$, its packing was saved explicitly. For low-value items, we save the first and last element entirely inside a bundle and on query of an item decide its membership in a bundle by comparing its density with those pivot elements.
- **Solution Value Query:** While the algorithm works with rounded values, we may set up the data structure of Section 3 to additionally store the actual values of items and enable prefix computation on the actual values. We can compute and store the actual solution value after an update by summing the actual values of packed candidates and determining the actual value of items in B using prefix computations while subtracting the values of discarded fractional bundles and items. On query, we return the stored solution value.
- **Single Knapsack Query:** Output the saved packing of all candidates packed in the knapsack. Then, in the respective density sorted data structure, iterate over items in bundles that were packed in the queried knapsack and output them. As above, this is possible since the first and last item of a bundle were saved during the update step.

- **Query Entire Solution:** Output saved packing of all candidates and iterate over items in packed bundles in the respective density sorted data structure as above.

Lemma B.5. *The query times of our algorithm are as follows.*
(i) Single item queries are answered in time $\mathcal{O}(\log \frac{m^2}{\varepsilon^6})$.
(ii) Solution value queries are answered in time $\mathcal{O}(1)$.
(iii) Queries of a single knapsack P_K are answered in time $\mathcal{O}(|P_K|)$.
(iv) Queries of the entire solution P are answered in time $\mathcal{O}(|P|)$.

Proof. (i): Since the packing of candidates is stored explicitly, each of the packed candidates can be output in time $\mathcal{O}(1)$. The part of the solution corresponding to low-value items is stored implicitly, by saving the correct guesses and the first and last items of each bundle. The latter are stored in a tree sorted by density first and item index second, as in the data structure that was used to compute the bundles. Also save a pointer to and from the respective adjoining bundles of these items. This preparation is done during an update. When a low-value item is queried, use these pivot items to determine whether it is contained in packed bundles and if so in which it lies. This takes time $\mathcal{O}(\log \frac{m^2}{\varepsilon^6})$.

(ii): The computations for this query are done during an update of the instance, with the update clearly dominating the runtime. Thus, on a query, the answer can be given in constant time.

(iii): As in (i), the packing of candidates in P_K can be output in time $\mathcal{O}(1)$. For low-value items we create, during an update, pointers from bundles to the first, i.e., densest, item contained in them. On a query, we then simply consider each bundle in P_K and iterate over the density sorted data structure used to find and output all items of the bundle.

(iv): We use the approach from (iii) on all knapsacks. □

C Proofs for Dynamic Linear Grouping

In this section, we give the technical details of the analysis of the dynamic linear grouping approach developed in Section 5.1. We start by analysing the approximation ratio, i.e., by formally proving Lemma 5.2. We restate it here for convenience.Recall that OPT is the optimal solution and $\text{OPT}_{\mathcal{T}}$ is the optimal solution attainable by packing item types $\mathcal{T}$ instead of items in J' and using $J \setminus J'$ without any changes.

Lemma 5.2. *Let* OPT *and* $\text{OPT}_{\mathcal{T}}$ *be as defined above. Then,* $v(\text{OPT}_{\mathcal{T}}) \geq \frac{(1-\varepsilon)(1-2\varepsilon)}{(1+\varepsilon)^2} v(\text{OPT})$.

The loss in the objective function due to rounding item values to natural powers of $(1+\varepsilon)$ is bounded by a factor of $(1-\varepsilon)$ by Lemma 3.1. As already pointed out, the analysis of the approximation ratio consists of three steps. In Lemma C.1 we show that the loss in the objective function value when restricting the items in J' to the value classes with $\bar{\ell} \leq \ell \leq \ell_{\max}$ is bounded by a factor of $(1-\varepsilon)$. If an optimal solution contains n_ℓ items of V'_ℓ, it is feasible to pack the n_ℓ *smallest* such items. Then, Lemma C.2 shows that we do not need n_ℓ exactly but it suffices to guess n_ℓ up to a factor of $(1+\varepsilon)$. Finally, in Lemma C.3, we argue that using the introduced dynamic linear grouping approach costs at most a factor $(1-2\varepsilon)$. In Lemma 5.3, we show that the number of item types within one value class is reduced to $\mathcal{O}(\frac{\log n'}{\varepsilon^2})$. In Lemma 5.4, we bound the running time of the algorithm.

Let $\mathcal{P}_1$ be the set of solutions that (i) may use all items in J'' and (ii) uses items in J' only of the value classes V_ℓ with $\bar{\ell} \leq \ell \leq \ell_{\max}$. Let OPT_1 be an optimal solution in $\mathcal{P}_1$. The following lemma bounds the value of OPT_1 in terms of OPT.

Lemma C.1. *Let* OPT_1 *be defined as above. Then,* $v(\text{OPT}_1) \geq (1-\varepsilon)v(\text{OPT})$.

Proof. Given $\ell_{\max}$, it follows that $v(\textsc{Opt}) \geq (1+\varepsilon)^{\ell_{\max}}$. As n' is an upper bound on the cardinality of $\textsc{Opt}'$, the items in the value classes with $l < \bar{\ell}$ contribute at most $n'-1$ items to $\textsc{Opt}'$ while the value of one item is bounded by $(1+\varepsilon)^{\bar{\ell}}$. Thus, the total value of items in $V_0, \ldots, V_{\bar{\ell}}$ contributing to $\textsc{Opt}'$ is bounded by

$$\begin{aligned}
n'(1+\varepsilon)^{\bar{\ell}} &= n'(1+\varepsilon)^{\ell_{\max} - \left\lceil \frac{\log n'/\varepsilon}{\log(1+\varepsilon)} \right\rceil} \\
&\leq n'(1+\varepsilon)^{\ell_{\max} - \frac{\log(n'/\varepsilon)}{\log(1+\varepsilon)}} \\
&= n'(1+\varepsilon)^{\ell_{\max}}(1+\varepsilon)^{-\log_{1+\varepsilon}(n'/\varepsilon)} \\
&= n'(1+\varepsilon)^{\ell_{\max}} \frac{\varepsilon}{n'} \\
&\leq \varepsilon(1+\varepsilon)^{\ell_{\max}} \\
&\leq \varepsilon v(\textsc{Opt}).
\end{aligned}$$

Let J_1 be the items in $\textsc{Opt}'$ restricted to the value classes with $\bar{\ell} \leq \ell \leq \ell_{\max}$. Clearly, J_1 and $\textsc{Opt}''$ can be feasibly packed. Hence,

$$v(\textsc{Opt}_1) \geq v(J_1) + v(\textsc{Opt}'') \geq v(\textsc{Opt}') - \varepsilon v(\textsc{Opt}) + v(\textsc{Opt}'') \geq (1-\varepsilon)v(\textsc{Opt}).$$

□

From now on, we only consider packings in $\mathcal{P}_1$, i.e., we restrict to the value classes V_ℓ with $\bar{\ell} \leq l \leq \ell_{\max}$ for the items in J'. Let V_ℓ be a value class contributing to $\textsc{Opt}_1'$. As explained above, knowing $n_\ell = |V_\ell \cap \textsc{Opt}_1'|$ would be sufficient to determine the items of V_ℓ' contributing to $\textsc{Opt}_1$, i.e., to determine $V_\ell' \cap \textsc{Opt}_1$. In the following lemma we show that we can additionally assume that $n_\ell = (1+\varepsilon)^{k_\ell}$ for some $k_\ell \in \mathbb{N}_0$. To this end, let $\mathcal{P}_2$ contain all the packings in $\mathcal{P}_1$ where the number of big items of each value class V_ℓ is a natural power of $(1+\varepsilon)$. Let $\textsc{Opt}_2$ be an optimal packing in $\mathcal{P}_2$.

Lemma C.2. *Let* $\textsc{Opt}_2$ *be as defined above. Then,* $v(\textsc{Opt}_2) \geq \frac{1}{(1+\varepsilon)} v(\textsc{Opt}_1)$.

Proof. Consider $\textsc{Opt}_1$, the optimal packing in $\mathcal{P}_1$. We set $\textsc{Opt}_1' := \textsc{Opt}_1 \cap J'$ and $\textsc{Opt}_1'' := \textsc{Opt}_1 \setminus J' = \textsc{Opt}_1 \cap J''$. We construct a feasible packing in $\mathcal{P}_2$ that achieves the desired value of $\frac{1}{(1+\varepsilon)} v(\textsc{Opt}_1)$.

Let J_2 be the subset of $\textsc{Opt}_1'$ where each value class V_ℓ' is restricted to the smallest $(1+\varepsilon)^{\lfloor \log_{1+\varepsilon} n_\ell \rfloor}$ items in V_ℓ' if $V_\ell \cap \textsc{Opt}_1' \neq \emptyset$.

Fix one value class V_ℓ with $V_\ell \cap \textsc{Opt}_1' \neq \emptyset$. Restricting to the first $(1+\varepsilon)^{\lfloor \log_{1+\varepsilon} n_\ell \rfloor}$ items in $V_\ell \cap \textsc{Opt}_1'$ implies

$$\begin{aligned}
v(V_\ell \cap J_2) &= (1+\varepsilon)^{\lfloor \log_{1+\varepsilon} n_\ell \rfloor}(1+\varepsilon)^\ell \\
&\geq \frac{1}{1+\varepsilon}(1+\varepsilon)^{\log_{1+\varepsilon} n_\ell}(1+\varepsilon)^\ell \\
&= \frac{1}{1+\varepsilon}(1+\varepsilon)^\ell n_\ell \\
&= \frac{1}{1+\varepsilon} v(V_\ell \cap \textsc{Opt}_1').
\end{aligned}$$

Clearly, $J_2 \cup \textsc{Opt}_1''$ is a feasible packing in $\mathcal{P}_2$. Observe that $v(\textsc{Opt}_1') = \sum_{\ell=\bar{\ell}}^{\ell_{\max}} v(V_\ell \cap \textsc{Opt}_1')$. Hence,

$$v(\textsc{Opt}_2) \geq v(J_2) + v(\textsc{Opt}_1'') \geq \frac{1}{1+\varepsilon} v(\textsc{Opt}_1') + v(\textsc{Opt}_1'') \geq \frac{1}{1+\varepsilon} v(\textsc{Opt}_1).$$

□

From now on, we only consider packings in $\mathcal{P}_2$. This means, we restrict the items in J' to value classes V'_ℓ with $\bar{\ell} \le \ell \le \ell_{\max}$ and assume that $n_\ell = (1+\varepsilon)^{k_\ell}$ for $n_\ell \in \mathbb{N}_0$ or $n_\ell = 0$. Even with n_ℓ being of the form $(1+\varepsilon)^{k_\ell}$, guessing the exponent for each value class V'_ℓ independently is intractable in time polynomial in $\log n$ and $\frac{1}{\varepsilon}$. To resolve this, the dynamic linear grouping creates groups that take into account all possible guesses of n_ℓ. This rounding is done for each value class individually and results in item types $\mathcal{T}_\ell$ for the set V'_ℓ. Let $\mathcal{P}_\mathcal{T}$ be the set of all feasible packings of items in $\mathcal{T}_\ell$ for $\bar{\ell} \le \ell \le \ell_{\max}$ and any subset of items in J''. That is, instead of the original items in J' the packings in $\mathcal{P}_\mathcal{T}$ pack the corresponding item types. Note that packings in $\mathcal{P}_\mathcal{T}$ are not forced to pack natural powers of $(1+\varepsilon)$ many items per value class. Let $\textsc{Opt}_\mathcal{T}$ be the optimal solution in $\mathcal{P}_\mathcal{T}$. The next lemma shows that $v(\textsc{Opt}_\mathcal{T})$ is at most a factor $(1-2\varepsilon)$ less than $v(\textsc{Opt}_2)$, the optimal solution in $\mathcal{P}_2$.

Lemma C.3. *Let* $\textsc{Opt}_3$ *be defined as above. Then,* $v(\textsc{Opt}_\mathcal{T}) \ge (1-2\varepsilon)v(\textsc{Opt}_2)$.

Proof. We construct a feasible packing J_3 in $\mathcal{P}_\mathcal{T}$ based on the optimal packing $\textsc{Opt}_2$. Let $\textsc{Opt}'_2 := J' \cap \textsc{Opt}_2$ and $\textsc{Opt}''_2 := J'' \cap \textsc{Opt}$. We let $J''_3 := \textsc{Opt}''_2$ be the items of J'' in our new packing J_3. These items will be packed exactly where they are packed in $\textsc{Opt}_2$. For items in J', we consider each value class $V_\ell \cap \textsc{Opt}'_2$ individually and carefully construct the set $J_{3,l}$, the items of V'_ℓ contributing to J_3. Then, we show that the items in $J_{\ell,3}$ can be packed into the knapsacks where the items in $V_\ell \cap \textsc{Opt}'_2$ are placed while ensuring that $v(J_{\ell,3}) \ge (1-2\varepsilon)v(V_\ell \cap \textsc{Opt}'_2)$.

If $V_\ell \cap \textsc{Opt}'_2 = \emptyset$, we set $J_{\ell,3} = \emptyset$. Then, both requirements are trivially satisfied. Consider the case where $|V'_\ell \cap \textsc{Opt}'_2| \le \frac{1}{\varepsilon}$. Then, we set $J_{\ell,3} := V'_\ell \cap \textsc{Opt}'_2$. Clearly, $v(J_{\ell,3}) \ge (1-2\varepsilon)v(V_\ell \cap \textsc{Opt}'_2)$.

For packing $J_{\ell,3}$, we observe that $\mathcal{T}_\ell$ actually contains the smallest $\frac{1}{\varepsilon}$ items as item types. Hence, their sizes are not affected by the rounding procedure and whenever $\textsc{Opt}_2$ packs one of these items, we can pack the same item into the same knapsack.

Let ℓ be a value class with $n_\ell := |V'_\ell \cap \textsc{Opt}'_2| > \frac{1}{\varepsilon}$. Let $G_1(n_\ell), \ldots, G_{1/\varepsilon}(n_\ell)$ be the corresponding $\frac{1}{\varepsilon}$ groups of $\lfloor \varepsilon n_\ell \rfloor$ or $\lceil \varepsilon n_\ell \rceil$ many items created by the (traditional) linear grouping for n_ℓ. We set $J_{\ell,3} = G_1(n_\ell) \cup \ldots \cup G_{1/\varepsilon-1}(n_\ell)$. As $v(G_{1/\varepsilon}(n_\ell)) = \lceil \varepsilon n_\ell \rceil (1+\varepsilon)^\ell \le 2\varepsilon n_\ell (1+\varepsilon)^\ell = 2\varepsilon v_{\ell,2}$, we have $v(J_{\ell,3}) \ge (1-2\varepsilon)v(V'_\ell \cap \textsc{Opt}'_2)$. For packing these items, we observe that the item types created by our algorithm are a refinement of $G_1(n_\ell), \ldots, G_{1/\varepsilon}(n_\ell)$. As the dynamic linear grouping ensures $|G_{1/\varepsilon}(n_\ell)| \ge |G_{1/\varepsilon-1}(n_\ell)| \ge \ldots \ge |G_1(n_\ell)|$ and that the item sizes are increasing in the group index, we can pack the items of group $G_k(n_\ell)$ where $\textsc{Opt}_2$ packs the items of group $G_{k+1}(n_\ell)$ for $1 \le k < \frac{1}{\varepsilon}$.

We conclude

$$v(\textsc{Opt}_\mathcal{T}) \ge v(J_3) + v(\textsc{Opt}''_2) \ge (1-2\varepsilon)v(\textsc{Opt}'_2) + v(\textsc{Opt}''_2) \ge (1-2\varepsilon)v(\textsc{Opt}_2).$$

□

Next, we formally prove the bound on the running time, i.e., Lemma 5.4.

Lemma 5.4. *For a given guess* $\ell_{\max}$*, the set* $\mathcal{T}^{(\ell_{\max})}$ *can be determined in time* $\mathcal{O}(\frac{\log^4 n'}{\varepsilon^4})$.

Proof. Remember that n' is an upper bound on the number of items in J' in any feasible solution. Observe that the boundaries of the linear grouping created by the algorithm per value class are actually independent of the value class and only refer to some kth item in class V_ℓ. Hence, the algorithm first computes the different indices needed in this round. We denote the set of these indices by $I' = \{j_1, \ldots\}$ sorted in an increasing manner. There are at most $\lfloor \log_{1+\varepsilon} n' \rfloor$ many possibilities for n_ℓ. Thus, the algorithm needs to compute at most $\frac{1}{\varepsilon}(\log_{1+\varepsilon} n' + 1)$ many different indices. This means that these indices can be computed and stored in time $\mathcal{O}(\frac{\log n'}{\varepsilon^2})$ while each index is bounded by n.

Given the guess $\ell_{\max}$ and $\bar{\ell}$, fix a value class V_ℓ with $\bar{\ell} \le l \le \ell_{\max}$. We want to bound the time the algorithm needs to transform the big items in V_ℓ into the modified item set T_ℓ. We will ensure that the dynamic algorithms in the following sections maintain a balanced binary search for each value class V_ℓ that stores the items in J' sorted by increasing size. Hence, the sizes of the items corresponding to J' can be accessed in time $\mathcal{O}(\frac{\log^3 n'}{\varepsilon^2})$. These sizes correspond to the item size s_t for an item type $t \in \mathcal{T}_\ell$. Given an item type $t \in \mathcal{T}_\ell$, $n_t = j_t - j_{t-1}$, which can again be pre-computed independently of the value class. Thus, $\mathcal{T}_\ell$ can be computed in time $\mathcal{O}(\frac{\log^3 n'}{\varepsilon^2})$.

As there are $\mathcal{O}(\frac{\log n'}{\varepsilon^2})$ many value classes that need to be considered for a given guess $\ell_{\max}$, calculating the set $\mathcal{T}^{(\ell_{\max})}$ needs $\mathcal{O}(\frac{\log^4 n'}{\varepsilon^4})$ many computation steps. □

C.1 Integrally Packing Fractional Solutions

One of the main ingredients to the dynamic algorithms in this section is a configuration ILP. As solving general ILPs is NP-hard, in a first step, we relax the integrality constraints and accept fractional solutions before rounding the obtained solution to an integral one. The first lemma of this section describes how to obtain an integral solution with slightly more knapsacks given a fractional solution to a certain class of packing ILPs. Even after rounding, the configuration ILPs only take care of integrally packing big items, i.e., items with $s_j \geq \varepsilon S_i$. Therefore, the second lemma focuses on packing small items integrally given an integral packing of big items that reserves enough space for packing these items fractionally using resource augmentation.

We consider a packing problem of items into a given set of knapsacks K with capacities S_k and multiplicities m_k. The objective is to maximize the total value without violating any capacity constraint. Each item j has a certain type t, i.e., value $v_j = v_t$ and size $s_j = s_t$, and in total there are n_t items of type t. Items can either be packed as single items or as part of configurations. A configuration c has value $v_c = \sum_{j \in c} v_j$ and size $s_c = \sum_{j \in c} s_j$. Then, the set E represents the items and the configurations that we are allowed to use for packing. Without loss of generality, we assume that for each element $e \in E$ there exists at least one knapsack i where this element can be stored.

Let $0 \leq \beta \leq 1$ and $s \geq 0$. Intuitively, later we will choose $\beta = 1 - O(\varepsilon)$ since we will leave an $O(\epsilon)$-fraction of the knapsacks unused. Consider the packing ILP for the above described problem with variables $z_{e,k}$ that may additionally contain constraints of the form

$$\sum_{e \in E, k \in K'} s_e z_{e,k} \leq \beta \sum_{k \in K'} m_k \, S_k - s \text{ and } \sum_{e \in E, k \in K'} z_{e,k} \leq \beta \sum_{k \in K'} m_k \, ,$$

i.e., the elements assigned to a subset of knapsacks K' do not violate the total capacity of a β-fraction of the knapsacks in K' while reserving a space of size s and use at most a β-fraction of the available knapsacks. The configuration ILPs used for multiple identical knapsacks and multiple different knapsacks with resource augmentation fall into this class of problems.

Let $v(z)$ be the value attained by a certain solution z and let $n(z)$ be the number of non-zero variables of z. The high-level idea of the proof of the following lemma is to round down each non-zero variable $z_{e,k}$ and pack the corresponding elements as described by $z_{e,k}$. For packing enough value, we additionally place one extra element e into the knapsacks given by resource augmentation for each variable $z_{e,k}$ that was subjected to rounding.

More precisely, for each element e and each knapsack type k, let $\bar{z}'_{e,k} := \lfloor z_{e,k} \rfloor$ and $\bar{z}''_{e,k} := \lceil z_{e,k} - \bar{z}'_{e,k} \rceil$. Note that $\bar{z}' + \bar{z}''$ may require more items of a certain type than are available. Hence, for each item type t that is now packed more than n_t times, we reduce the number of t in $\bar{z}' + \bar{z}''$ by either adapting the chosen configurations if t is packed in a configuration or by decreasing the variables of type $z_{t,k}$ if items of type t are packed as single items in knapsacks of type k. Let z' and z'' denote the solution obtained by this transformation. For some elements e, $z'_{e,k} + z''_{e,k}$ may now pack more or less elements than $z_{e,k}$ due to the just described reduction of items.

Lemma C.4. *Any fractional solution z to the above described packing ILP can be rounded to an integral solution with value at least $v(z)$ using at most $n(z)$ additional knapsacks.*

Proof. Consider a particular item type t. If $\bar{z}' + \bar{z}''$ packs at most n_t items of this type, then the value achieved by z for this particular item type is upper bounded by the value achieved by $z' + z''$. If an item type was subjected to the modification, then $z' + z''$ packs exactly n_t items of this type while z packs at most n_t items. This implies that $v(z' + z'') \geq v(z)$.

It remains to show how to pack $\bar{z}' + \bar{z}''$ (and, thus, $z' + z''$) into the knapsacks defined by K and potentially $n(z)$ additional knapsack. Clearly, $\bar{z}'$ can be packed exactly as z was packed. If $z_{e,k} = 0$ for $e \in E$ and $k \in K$, then $\bar{z}'_{e,r} = 0$. Hence, the number of non-zero entries in $\bar{z}''$ is bounded by $n(z)$. Consider one element $e \in E$ and a knapsack type k with $\bar{z}''_{e,k} = 1$ and let k' be a knapsack where e fits. Pack e into k'.

As reducing the number of packed items of a certain type only decreases the size of the corresponding configuration or only decreases the number of individually packed elements, $z' + z''$ can be packed exactly as described for $\bar{z}' + \bar{z}''$. Then, we need at most $n(z)$ extra knapsacks to pack z'' which concludes the proof. □

After having successfully rounded an ILP solution, we explain how to pack small items, i.e., items with $s_j < \varepsilon S_i$, using resource augmentation given an integral packing of big items. More precisely, let K be a set of knapsacks and let $J'_S \subset J_S$ be a subset of items that are small with respect to any knapsack in K. Let $J'_B \subset J_B$ be a set of big items admitting an integral packing into $m = |K|$ knapsacks that preserves a space of at least $s(J'_S)$ in these m knapsacks. We develop a procedure to extend this packing to an integral packing of all items $J'_B \cup J'_S$ in $(1 + \varepsilon)m$ knapsacks where the εm additional knapsacks can be chosen to have the smallest capacity of knapsacks in K.

We use a packing approach similar to NEXT FIT for the problem BIN PACKING. That is, consider an arbitrary order of the small items and an arbitrary order of the knapsacks filled with big items. If the current small item j still fits into the currently open knapsack, we place it there and decrease the remaining capacity accordingly. If it does not fit anymore, we pack this item as "cut" item into the next empty slot of the εm additional knapsacks, close the current knapsack and open the next one for packing small items.

Lemma C.5. *Let K, J'_S, and J'_B be as defined above. The procedure described above feasibly packs all items $J'_B \cup J'_S$ in $(1+\varepsilon)m$ knapsacks where the εm additional knapsacks can be chosen to have the smallest capacity of knapsacks in K.*

Proof. Clearly, the packing created by the procedure is integral and feasible. It remains to bound the number of additional knapsacks. Observe that each item that we packed into the resource augmentation while an original knapsack was still available, implied the closing of the current knapsack and the opening of a new one. Hence, for each original knapsack at most one small item was placed into the additional knapsacks. Thus, at most m small items are packed into the additional knapsacks. As at least $\frac{1}{\varepsilon}$ items fit into one additional knapsack, we only need εm extra knapsacks for such items. We complete the proof by showing that all items in J'_S are indeed packed. Assume that there is a small item j left *after* all knapsacks, original and opened extra knapsacks, were closed while packing small items. As a knapsack is only closed if the current small item does not fit anymore, this implies that the volume of all items that are packed so far have a total volume at least as large as the total capacity of knapsacks in K. Hence, the total volume of all items in $J'_B \cup J'_S$ is strictly larger than the total capacity of knapsacks in K as j is left unpacked after all knapsacks have been closed. This contradicts the assumption imposed on J'_B and on J'_S. □

D Proofs for Identical Knapsacks

In this section we give the technical details of some of the lemmas used in Section 5.2. We start by proving Lemma 5.7. Recall that v^*_{ILP} refers to an optimal, integral solution to ILP (P) and that $\textsc{Opt}_{\mathcal{T}}$ is an optimal solution to the current instance when packing item types $\mathcal{T}$ instead of the big items J_B. Further, P is the maximal prefix of small items (ordered by non-decreasing density) with $v(P) < v_S$ and j is the densest small item not in P.

Lemma 5.7. *Let v^*_{ILP} and $\textsc{Opt}_{\mathcal{T}}$ be defined as above. There are v_S and s_S with $v^*_{ILP} + v_S \geq \frac{1-3\varepsilon}{1+\varepsilon} v(\textsc{Opt}_{\mathcal{T}})$. Moreover, for P and j as defined above, $v(P) + v_j \geq v_S$.*

Proof. Let $\textsc{Opt}_{B,\mathcal{T}} := \textsc{Opt}_{\mathcal{T}} \cap J_B$ and $\textsc{Opt}_{S,\mathcal{T}} := \textsc{Opt}_{\mathcal{T}} \cap J_S$. We construct again a candidate set J_{ILP} of items that are feasible for (P) and obtain a value of at least $(1-3\varepsilon)v(\textsc{Opt}_{B,\mathcal{T}})$. To this end, take the optimal packing for the items in $\textsc{Opt}_{\mathcal{T}}$ and consider the $(1-3\varepsilon)m$ most valuable knapsacks in this packing. Let $J_{B,\mathcal{T}}$ and $J_{S,\mathcal{T}}$ consist of the big and small, respectively, items in these knapsacks. Then, $v(J_{B,\mathcal{T}}) + v(J_{S,\mathcal{T}}) \geq (1-3\varepsilon)v(\textsc{Opt}_{\mathcal{T}})$.

Create the variable values y_c corresponding to the number of times configuration c is used by the items in $J_{B,\mathcal{T}}$. As $J_{B,\mathcal{T}} \cup J_{S,\mathcal{T}}$ can be feasibly packed into $(1-3\varepsilon)m$ knapsacks, we have

$$\sum_{c \in \mathcal{C}} y_c \leq (1-3\varepsilon)m$$

and

$$\sum_{c \in \mathcal{C}} y_c s_c + s(J_{S,3}) \leq (1-3\varepsilon)Sm.$$

As we guess the value of the small items in the dynamic algorithm up to factors of $(1+\varepsilon)$ there will be one guess v_S satisfying $v_S \leq v(J_{S,\mathcal{T}}) \leq (1+\varepsilon)v_S$. Let P be the maximal prefix of small items with $v(P) < v_S$ and let j be the densest small item not in P. Then, $v(P) + v_j \geq v_S \geq \frac{1}{1+\varepsilon} v(J_{S,\mathcal{T}})$.

As P contains the densest small items, this implies $s_S := s(P) \leq s(J_{S,\mathcal{T}})$. Thus,

$$\sum_{c \in \mathcal{C}} y_c s_c \leq (1-3\varepsilon)Sm - s(J_{S,\mathcal{T}}) \leq (1-3\varepsilon)Sm - s_S.$$

Hence, the just created y_c are feasible for the ILP with the guess s_S and

$$\begin{aligned} v^*_{\text{ILP}} + v(P) + v_j &\geq v(J_{B,\mathcal{T}}) + \frac{1}{1+\varepsilon} v(J_{S,\mathcal{T}}) \\ &\geq (1-3\varepsilon)\left(v(\textsc{Opt}_{B,\mathcal{T}}) + \frac{1}{1+\varepsilon} v(\textsc{Opt}_{S,\mathcal{T}})\right) \geq \frac{1-3\varepsilon}{1+\varepsilon} v(\textsc{Opt}_{\mathcal{T}}). \end{aligned}$$

□

Proof of Lemma 5.8 In this part, we provide the full proof of our approach to solving the LP relaxation of the configuration ILP when m satisfies $\frac{16}{\varepsilon^7} \log^2 n \leq m$. In the following, we abuse notation and also refer to the LP relaxation of (P) by (P):

$$
\begin{array}{llll}
\max & \sum_{c\in\mathcal{C}} y_c v_c & & \\
\text{subject to} & \sum_{c\in\mathcal{C}} y_c s_c & \le & (1-3\varepsilon)Sm - s_S \\
& \sum_{c\in\mathcal{C}} y_c & \le & (1-3\varepsilon)m \\
& \sum_{c\in\mathcal{C}} y_c n_{tc} & \le & n_t \qquad \text{for all } t\in\mathcal{T}^{(l_{\max})} \\
& y_c & \in & \ge 0 \qquad \text{for all } c\in\mathcal{C}
\end{array} \tag{P}
$$

Let γ and β be the dual variables of the capacity constraint and the number of knapsacks constraint, respectively. We set $\mathcal{T} := \mathcal{T}^{(\ell_{\max})}$ for simplicity. Let α_t for $t \in \mathcal{T}$ be the dual variables of the constraint ensuring that only n_t items of type t are packed. Then, the dual is given by the following linear program.

$$
\begin{array}{llll}
\min & (1-3\varepsilon)m\beta + ((1-3\varepsilon)Sm - s_S)\gamma + \sum_{t\in\mathcal{T}} n_t\alpha_t & & \\
\text{subject to} & \beta + s_c\gamma + \sum_{t\in T} \alpha_t n_{tc} & \ge & v_c \quad \text{for all } c\in\mathcal{C} \\
& \alpha_t & \ge & 0 \quad \text{for all } t\in\mathcal{T} \\
& \beta,\gamma & \ge & 0.
\end{array} \tag{D}
$$

As discussed above, for applying the Ellipsoid method we need to solve the separation problem efficiently. The separation problem decides if the current solution $(\alpha^*, \beta^*, \gamma^*)$ is feasible or finds a violated constraint. As verifying the first constraint of (D) corresponds to solving a KNAPSACK problem, we do not expect to optimally solve the separation problem in time polynomial in $\log n$ and $\frac{1}{\varepsilon}$. Instead, we apply a dynamic program (DP) for the single knapsack problem after restricting the item set further and rounding the item values as follows.

Let $\bar{v}_t := v_t - \alpha_t^* - \gamma^* s_t$ for $t \in \mathcal{T}$. If there exists an item type with $\bar{v}_t > \beta^*$, we return the configuration using only this item. Otherwise, we define $\tilde{v}_t := \left\lfloor \frac{\bar{v}_t}{\varepsilon^4\beta^*} \right\rfloor \cdot \varepsilon^4\beta^*$. By running the dynamic program for the KNAPSACK problem on the item set $\mathcal{T}$ with multiplicities $\min\{\frac{1}{\varepsilon}, n_t\}$ and values $\tilde{v}_t$, we obtain a solution x^* where x_t^* indicates how often item type t is packed. If $\sum_{t\in\mathcal{T}} x_t^* \tilde{v}_t > \beta^*$, we return the configuration defined by x^* as separating hyperplane. Otherwise, we return DECLARED FEASIBLE for the current solution.

The next lemma shows that this algorithm approximately solves the separation problem by either correctly declaring infeasibility or by finding a solution that is almost feasible for (D). The slight infeasibility for the dual problem translates to a slight decrease in the optimal objective function value of the primal problem. In the proof we use that x^* is optimal for the rounded values $\tilde{v}_t$ to show that $(\alpha^*, \beta^*, \gamma^*)$ is almost feasible if $\sum_{t\in\mathcal{T}} x_t^* \tilde{v}_t \le \beta^*$. Noticing that $\bar{v}_t \ge \tilde{v}_t$ then concludes the proof.

Lemma D.1. *Given $(\alpha^*, \beta^*, \gamma^*)$, there is an algorithm with running time $\mathcal{O}\left(\frac{\log^2 n}{\varepsilon^{10}}\right)$ which either finds a configuration $c \in \mathcal{C}$ such that $\beta^* + s_c\gamma^* + \sum_{t\in\mathcal{T}} \alpha_t^* n_{tc} < v_c$ or guarantees that $\beta^* + s_c\gamma^* + \sum_{t\in\mathcal{T}} \alpha_t^* n_{tc} \ge (1-\varepsilon)v_c$ holds for all $c \in \mathcal{C}$.*

Proof. Fix a configuration c and recall that $s_c = \sum_{t\in\mathcal{T}} n_{tc}s_t$ and $v_c = \sum_{t\in\mathcal{T}} n_{tc}v_t$. Then, checking $\beta^* + s_c\gamma^* + \sum_{t\in\mathcal{T}} \alpha_t^* n_{tc} \ge v_c$ for all configurations $c \in \mathcal{C}$ is equivalent to showing $\max_{c\in\mathcal{C}} \sum_{t\in\mathcal{T}} (v_t - \alpha_t^* - \gamma^* s_t) n_{tc} \le \beta^*$. This problem translates to solving the following ILP and comparing its objective function value to β^*.

$$\begin{array}{llll} \max & \sum_{t\in\mathcal{T}} (v_t - \alpha_t^* - \gamma^* s_t) x_t & & \\ \text{s.t.} & \sum_{t\in\mathcal{T}} s_t x_t & \leq & S \\ & x_t & \leq & n_t \quad \text{for all } t \in \mathcal{T} \\ & x_t & \in & \mathbb{Z}_{\geq 0} \end{array} \tag{S}$$

This ILP is itself a (single) KNAPSACK problem. Hence, the solution x^* found by the algorithm is indeed feasible for (S).

We start by bounding the running time of the algorithm. Recall that, for each $t \in \mathcal{T}$, $\bar{v}_t := v_t - \alpha_t^* - \gamma^* s_t$ and $\tilde{v}_t := \lfloor \frac{\bar{v}_t}{\varepsilon^4 \beta^*} \rfloor \cdot \varepsilon^4 \beta^*$. Observe that $\mathcal{T}$ only contains big items. Hence, it suffices to consider $\min\{n_t, \frac{1}{\varepsilon}\}$ items per value class in the DP. It can be checked in time $\mathcal{O}(\frac{\log^2 n}{\varepsilon^4})$, if $\bar{v}_t \leq \beta^*$ is violated for one $t \in \mathcal{T}$. Otherwise, $\tilde{v}_t \leq \bar{v}_t$ and $\bar{v}_t - \tilde{v}_t \leq \varepsilon^4 \beta^*$ hold. Thus, the running time of the DP is bounded by $\mathcal{O}\left(\frac{|\mathcal{T}|^2}{\varepsilon^6}\right) = \mathcal{O}\left(\frac{\log^2 n}{\varepsilon^{10}}\right)$ [38].

It remains to show that the solution x^* either defines a configuration with $\beta^* + s_c\gamma^* + \sum_{t\in\mathcal{T}} \alpha_t^* n_{tc} < v_c$ or ensures that $\beta^* + s_c\gamma^* + \sum_{t\in\mathcal{T}} \alpha_t^* n_{tc} \geq (1-\varepsilon) v_c$ holds for all $c \in \mathcal{C}$. If $\sum_{t\in\mathcal{T}} x_t^* \tilde{v}_t > \beta^*$, it holds

$$\sum_{t\in\mathcal{T}} x_t^* \bar{v}_t \geq \sum_{t\in\mathcal{T}} x_t^* \tilde{v}_t > \beta^*$$

and, thus, x^* defines a separating hyperplane.

Suppose now that $\sum_{t\in\mathcal{T}} x_t^* \tilde{v}_t \leq \beta^*$. We assume for the sake of contradiction that there is a configuration c', defined by packing x_t items of type t, such that

$$\sum_{t\in\mathcal{T}} x_t \big((1-\varepsilon) v_t - \alpha_t^* - \gamma^* s_t \big) > \beta^*.$$

As $\mathcal{T}$ contains only big item types, we have that $\sum_{t\in\mathcal{T}} x_t \leq \frac{1}{\varepsilon}$. This implies that there exists at least one item type t' in $\mathcal{T}$ with $x_{t'} \geq 1$ and $(1-\varepsilon) v_{t'} - \alpha_{t'}^* - \gamma^* s_{t'} \geq \varepsilon \beta^*$. Moreover,

$$\bar{v}_t = v_t - \alpha_t^* - \gamma^* s_t \geq (1-\varepsilon) v_t - \alpha_t^* - \gamma^* s_t$$

holds for all item types $t \in \mathcal{T}$. This implies for t' that $\bar{v}_{t'} \geq \varepsilon \beta^*$. Hence,

$$\sum_{t\in\mathcal{T}} x_t \bar{v}_t \geq \varepsilon x_{t'} \bar{v}_{t'} + \sum_{t\in\mathcal{T}} x_t \big((1-\varepsilon) v_t - \alpha_t^* - \gamma^* s_t \big) > \varepsilon v_{t'} + \beta^* \geq (1+\varepsilon^2) \beta^*.$$

By definition of $\tilde{v}$, we have $\bar{v}_t - \tilde{v}_T \leq \varepsilon^4 \beta^*$ and $\sum_{t\in\mathcal{T}} x_t (\bar{v}_t - \tilde{v}_t) \leq \varepsilon^3 \beta^*$. This implies

$$\sum_{t\in\mathcal{T}} x_t \tilde{v}_t = \sum_{t\in\mathcal{T}} x_t \tilde{v}_t + \sum_{t\in\mathcal{T}} x_t (\bar{v}_t - \tilde{v}_t) > (1+\varepsilon^2)\beta^* - \varepsilon^3 \beta^* \geq \beta^*,$$

where the last inequality follows from $\varepsilon \leq 1$. By construction of the DP, x^* is the optimal solution for the values $\tilde{v}$ and achieves a total value less than or equal to β^*. Hence,

$$\beta^* \geq \sum_{t\in\mathcal{T}} x_t^* \tilde{v}_t \geq \sum_{t\in\mathcal{T}} x_t \tilde{v}_t > \beta^*;$$

a contradiction. □

We now present the proof of Lemma 5.8, which we restate for convenience.

Lemma 5.8. *Let* $U = \max\{Sm, nv_{\max}\}$. *Then, there is an algorithm that finds a feasible solution for the LP relaxation of* (P) *with value at least* $\frac{1-\varepsilon}{1+\varepsilon} v_{LP}$ *with running time bounded by* $\left(\frac{\log U}{\varepsilon}\right)^{\mathcal{O}(1)}$.

Proof. As discussed above, the high-level idea is to solve (D), the dual of (P), with the Ellipsoid method and to consider only the variables corresponding to constraints added by the Ellipsoid method for solving (P).

As (S) is part of the separation problem for (D), there is no efficient way to exactly solve the separation problem unless P = NP. Lemma D.1 provides us with a way to approximately solve the separation problem. As an approximately feasible solution for (D) cannot be directly used to determine the important variables in (P), we add an upper bound r on the objective function as a constraint to (D) and search for the largest r such that the Ellipsoid method returns infeasible. This implies that r is an upper bound on the objective function of (D) which in turn guarantees a *lower bound* on the objective function value of (P) by weak duality.

Of course, testing all possible values for r is intractable and we restrict the possible choices for r. Observe that $v_{\text{LP}} \in [v_{\max}, nv_{\max}]$ where v_{LP} is the optimal value of (P). Thus, for all $\lceil \log_{1+\varepsilon} v_{\max} \rceil \leq k \leq \lfloor \log_{1+\varepsilon}(nv_{\max}) \rfloor$, we use $r = (1+\varepsilon)^k$ as upper bound on the objective function. That is, we test if (D) extended by the objective function constraint $(1-3\varepsilon)m\beta + ((1-3\varepsilon)Sm - s_S)\gamma + \sum_{t\in\mathcal{T}} n_t\alpha_t \leq r$ is declared feasible by the Ellipsoid method with the approximate separation oracle for (S). We refer to the feasibility problem by (D_r).

For a given solution $(\alpha^*, \beta^*, \gamma^*)$ of (D_r) the separation problem asks for one of the two: either the affirmation that the point is feasible or a separating hyperplane that separates the point from any feasible point. It can be checked in time $\mathcal{O}(|\mathcal{T}|) \leq \mathcal{O}\big(\big(\frac{\log n}{\varepsilon^2}\big)^2\big)$ that α_t^*, β^*, and γ^* are non-negative. In case of a negative answer, the corresponding non-negativity constraint is a feasible separating hyperplane. Similarly, the objective function constraint $(1-3\varepsilon)\beta + (1-3\varepsilon)(Sm - s_k)\gamma + \sum_{t\in\mathcal{T}} n_t\alpha_t \leq r$ can be checked in time $\mathcal{O}(|\mathcal{T}|)$ where the numbers are bounded by $\log U$ and added as a new inequality if necessary. In case the non-negativity and objective function constraints are not violated, the separation problem is given by the knapsack problem in (S). The algorithm in Lemma D.1 either outputs a configuration that yields a valid separating hyperplane or declares $(\alpha^*, \beta^*, \gamma^*)$ feasible. That is $\beta^* + s_c\gamma^* + \sum_{t\in\mathcal{T}} \alpha_t^* n_{tc} \geq (1-\varepsilon)v_c$ holds for all $c \in \mathcal{C}$. This implies that $(\alpha^*, \beta^*, \gamma^*)$ is feasible for the following LP. (Note that we changed the right hand side of the constraints when compared to (D).)

$$
\begin{array}{llll}
\min & (1-3\varepsilon)\beta + ((1-3\varepsilon)Sm - s_S)\gamma + \sum_{t\in\mathcal{T}} n_t\alpha_t & & \\
\text{s.t.} & \beta + s_c\gamma + \sum_{t\in\mathcal{T}} \alpha_t n_{tc} \geq (1-\varepsilon)v_c & \text{for all } c \in \mathcal{C} & \quad (\text{D}^{(1-\varepsilon)}) \\
& \alpha_t \geq 0 & \text{for all } t \in \mathcal{T} & \\
& \beta, \gamma \geq 0 & &
\end{array}
$$

Let r^* be minimal such that (D_r) is declared feasible for $r = r^*$. Let $v_D^{(1-\varepsilon)}$ denote the optimal solution value of ($\text{D}^{(1-\varepsilon)}$). As $(\alpha^*, \beta^*, \gamma^*)$ is feasible and has an objective value of at most r^*, it follows $v_D^{(1-\varepsilon)} \leq r^*$.

Let $v^{(1-\varepsilon)}$ denote the optimal solution value of its dual, i.e., of the following LP.

$$\begin{array}{llll} \max & \sum_{c\in\mathcal{C}} y_c(1-\varepsilon)v_c & & \\ \text{subject to} & \sum_{c\in\mathcal{C}} y_c s_c & \leq & (1-3\varepsilon)Sm - s_S \\ & \sum_{c\in\mathcal{C}} y_c & \leq & (1-3\varepsilon)m \\ & \sum_{c\in\mathcal{C}} y_c n_{tc} & \leq & n_t \qquad \text{for all } t\in\mathcal{T}^{(l_{\max})} \\ & y_c & \in & \mathbb{Z}_{\geq 0} \qquad \text{for all } c\in\mathcal{C}. \end{array} \tag{P$^{(1-\varepsilon)}$}$$

Then, $y = 0$ is feasible for (P$^{(1-\varepsilon)}$) and by weak duality, we have

$$v^{(1-\varepsilon)} = v_D^{(1-\varepsilon)} \leq r^*.$$

Note that (P) and (P$^{(1-\varepsilon)}$) have the same feasible region and their objective functions only differ by the factor $(1-\varepsilon)$. This implies that

$$v_{\text{LP}} = \frac{v^{(1-\varepsilon)}}{1-\varepsilon} \leq \frac{r^*}{1-\varepsilon}. \tag{2}$$

Because of the relation between v_{LP} and r^* it suffices to find a feasible solution for (P) with objective function value close to r^* in order to prove the lemma.

To this end, let $\mathcal{C}_r$ be the configurations that correspond to the inequalities added by the Ellipsoid method while solving (D$_r$) for $r = \frac{r^*}{1+\varepsilon}$. Consider the problems (P) and (D) restricted to the variables y_c for $c \in \mathcal{C}_r$ and to the constraints for $c \in \mathcal{C}_r$, respectively, and denote these restricted LPs by (P′) and (D′). Let v' and v'_D be their respective optimal values.

It holds that $v'_D > r$ as the Ellipsoid method also returns infeasibility for (D′) when run on (D′) extended by the objective function constraint for r. As $y = 0$ is feasible for (P′) and $\alpha = 0$, $\beta = \max_{c\in\mathcal{C}_r} v_c$, and $\gamma = 0$ are feasible for (D′), their objective function values coincide, i.e., $v' = v'_D > r$. Since (P′) only has few variables, an optimal solution for (P′) can be found fast. Clearly, this solution is also feasible for (P) and achieves an objective function value

$$v' > \frac{r^*}{1+\varepsilon} \geq \frac{1-\varepsilon}{1+\varepsilon} v_{\text{LP}}$$

where we used Equation (2) for the last inequality.

It remains to show that the Ellipsoid method can be applied to the setting presented here and that the running time of the just described algorithm is indeed bounded by a polynomial in $\log n$, $\frac{1}{\varepsilon}$, and $\log U$. As the details are rather technical and contain no new insights, we refer to Appendix D.

Recall that U is an upper bound on the absolute values of the denominators and numerators appearing in (D), i.e., on Sm and $nv_{\max}$. Observe that by Lemma D.1, the separation oracle runs in time $\mathcal{O}\left(\frac{\log^2 n}{\varepsilon^{10}} + \frac{\log^2 n}{\varepsilon^4} \log S \log m\right)$; the additive term is due to checking non-negativity and the objective function constraint as first step of the separation problem. The number of iterations of the Ellipsoid method will be bounded by a polynomial in $\log U$ and $\tilde{n} \in \mathcal{O}\left(\frac{\log^2 n}{\varepsilon^4}\right)$. Here, $\tilde{n}$ is an upper bound on the number of variables in the problems (D$_r$) (and (D$^{(1-\varepsilon)}$)).

The feasible region of (D$_r$) is a subset of the feasible region of (D$^{(1-\varepsilon)}$), even when the objective function constraint is added to the latter LP. The Ellipsoid method usually is applied to full-dimensional, bounded polytopes that guarantee two bounds: (i) if the polytope is non-empty, then its volume is at least v and (ii) the polytope is contained in a ball of volume at most V. As shown in Chapter 8 of [7], these assumption can also be ensured and the parameters v and V can be chosen as polynomial functions of $\tilde{n}$ and U. As we cannot check feasibility of (D_r) directly, we choose the parameters v and V described in Chapter 8 of [7] for the problem ($D^{(1-\varepsilon)}$) extended by the objective function constraint for r. Then, after $N = \mathcal{O}(\tilde{n}\log(V/v))$ iterations, the modified Ellipsoid method either finds a feasible solution to ($D^{(1-\varepsilon)}$) with objective function value at most r or correctly declares (D_r) infeasible. Chapter 8 of [7] shows that the number of iterations N satisfies $N = \mathcal{O}(\tilde{n}^4 \log(\tilde{n}U))$ and that the overall running time is polynomially bounded in $\tilde{n}$ and $\log U$.

Hence, (P′), the problem (P) restricted to the constraints added by the Ellipsoid method, has at most N variables and, thus, any polynomial time algorithm for linear programs can be applied to (P′) to obtain an optimal solution. □

Answering Queries The remainder of this section is concerned with proving the results stated for answering queries. For convenience, we restate the corresponding lemmas.

Note that, throughout the course of the dynamic algorithm, we only implicitly store solutions. In the remainder of this section we explain how to answer the queries stated in Section 3 and bound the running times of the corresponding algorithms. We refer to the time frame between two updates as a *round* and introduce a counter τ that is increased after each update and denotes the current round. As our answers to queries have to stay consistent within a round, we *cache* existing query answers by aditionally storing a round $t(j)$ and a knapsack $k(j)$ for each item in the search tree for items where $t(j)$ stores the last round in which item j has been queried and $k(j)$ points to the knapsack of j in round $t(j)$. If j was NOT SELECTED in $t(j)$, we store this with $k(j) = 0$. Storing $t(j)$ is necessary since resetting the cached query answers after each update takes too much running time.

Let $\bar{y}_c$, $c \in \mathcal{C}$, be the packing for the big items in terms of the variables of the configuration ILP. During the run of the algorithm the set $\mathcal{C}' := \{c \in \mathcal{C} : \bar{y}_c \geq 1\}$ was constructed. We assume that this set is ordered in some way and stored in one list. In the following we use the position of $c \in \mathcal{C}'$ in that list as the index of c. For assigning $\bar{y}_c$ distinct knapsacks to configuration $c \in \mathcal{C}'$ we use the ordering of the configurations and map the knapsacks $\sum_{c'=1}^{c-1} \bar{y}_c + 1, \ldots, \sum_{c'=1}^{c} \bar{y}_{c'}$ to configuration c.

For small items, we store all items in a balanced binary search tree sorted by decreasing density. Let $1, \ldots, j^\star$ be the items selected by the implicit solution as guess for the size of small items in the current solution. Item $j^\star + 1$ is packed into its own knapsack. Any item $j \leq j^\star$ is either packed regularly into the empty space of a knapsack with a configuration or it is packed into a knapsack designated for packing "cut" small items. Therefore, we maintain two pointers: κ^r points to the next knapsack where a small item is supposed to go if it is packed *regularly* and κ^c points to the position where the next *cut* small item is packed. To determine if an item is packed regularly or as cut item, we store in ρ the remaining capacity of κ^r.

For each type t of big items, we maintain a pointer κ_t to the knapsack where the next queried item of type t is supposed to be packed. Moreover, the counter η_t stores how many slots κ_t still has for items of type t. Let $\bar{n}_t$ denote the number of items of type t belonging to solution $\bar{y}$. We will only pack the first $\bar{n}_t$ items of type t.

Answering Item Queries.

1) **Check cache.** Let τ be the current round and let j be the queried item. If $t(j) = \tau$, return $k(j)$.

2) **Answer queries for non-cached items.** Set $t(j) = \tau$. If $s_j \leq \varepsilon S$, item j is small. Otherwise, j is big.

   Small items. If $j > j^\star + 1$, return NOT SELECTED and set $k(j) = 0$.

   If $j = j^\star + 1$, return $k(j) = m$.

   Otherwise, determine if j is packed regularly or as cut item: If $s_j \leq \rho$, return $k(j) = \kappa^r$ and update ρ accordingly. Otherwise, return $k(j) = \kappa^c$. Increase κ^c to the next position for fractional items.

Increase κ^r by one and update ρ accordingly to reflect the empty space in κ^r.

Big items. Determine the value class V_ℓ of j. If $\ell < \ell_{\min}$, return NOT SELECTED. Otherwise, determine the item type t of j by retracing the steps of the dynamic linear grouping.

If j is not among the first $\bar{n}_t$ items of type t, return NOT SELECTED and set $k(j) = 0$.

Otherwise, return $k(j) = \kappa_t$ and decrease η_t by one. If $\eta_t = 0$, increase κ_t to the next knapsack for type t and update η_t accordingly. If no such knapsack exists, set $\kappa_t = 0$.

Answering the Solution Value Query.

1) **Value of small items.** Calculate $v_S = \sum_{j=1}^{j^\star+1} v_j$ with prefix computation.
2) **Value of big items.** For each item type t, calculate $v_{B,t}$ the value of the first $\bar{n}_t$ items of type t using prefix computation.
3) **Value.** Return $v_S + \sum_{t\in\mathcal{T}} v_{B,t}$.

Answering the Solution Query.

1) **Small items.** Query each item $j = 1, \dots j^\star + 1$ and return the solution.
2) **Big items.** For each type $t \in \mathcal{T}$, query the first $\bar{n}_t$ items and return the solution.

We prove the parts of the following lemmas individually.

Lemma 5.12. *The solution determined by the query algorithms is feasible and achieves the claimed total value. The query times of our algorithm are as follows.*

(i) Single item queries can be answered in time $\mathcal{O}\big(\max\big\{\log\frac{\log n}{\varepsilon}, \frac{1}{\varepsilon}\big\}\big)$
(ii) Solution value queries can be answered in time $\mathcal{O}(1)$
(iii) Queries of the entire solution P *are answered in time* $\mathcal{O}\big(|P|\max\big\{\log\frac{\log n}{\varepsilon}, \frac{1}{\varepsilon}\big\}\big)$.

Lemma D.2. *The solution determined by the query algorithms is feasible and achieves the claimed total value.*

Proof. By construction of $t(j)$ and $k(j)$, the answers to queries happening between two consecutive updates are consistent.

For small items, observe that $1, \dots, j^\star + 1$ are the densest small items in the current instance. By Lemma C.5, the packing obtained by our algorithms is feasible for these items. In Lemma 5.10 we argue that these items contribute enough value to our solution.

For big items, the algorithms correctly pack the first $\bar{n}_t$ items of type t. A knapsack with configuration $c \in \mathcal{C}'$ correctly obtains $n_{c,t}$ items of type t. Moreover, each configuration $c \in \mathcal{C}'$ gets assigned $\bar{y}_c$ knapsacks. Hence, the algorithms pack exactly the number of big items as dictated by the implicit solution $\bar{y}$. □

Lemma D.3. *The data structures for big items can be generated in time* $\mathcal{O}\big(\frac{|\mathcal{C}'|}{\varepsilon}\big)$. *Queries for big items can be answered in time* $\mathcal{O}\big(\log\frac{\log n}{\varepsilon}\big)$.

Proof. We assume that $\mathcal{C}'$ is already stored in some list. We start by formally mapping knapsacks to configurations. To this end, we create a list α where $\alpha_c = \sum_{c'=1}^{c-1} \bar{y}_c$ is the first knapsack with configuration $c \in \mathcal{C}'$. Using $\alpha_c = \alpha_{c-1} + \bar{y}_{c-1}$, we can compute these values in constant time while the appearing numbers are bounded by m. Hence, by iterating once through $\mathcal{C}'$, this list can be generated in $\mathcal{O}(|\mathcal{C}'|)$. By definition, the created lists are already ordered by increasing α_c.

We start by recomputing the indices needed for the dynamic linear grouping approach. For each value class V_ℓ with $\bar{\ell} \leq \ell \leq \max$, we access the items corresponding to the boundaries of the item types $\mathcal{T}_\ell$ in order to obtain the item types $\mathcal{T}_\ell$. By construction, these types are already ordered by non-decreasing size s_t. By Lemma 5.4, these item types can be computed in time $\mathcal{O}\big(\frac{\log^4 n}{\varepsilon^4}\big)$ and stored in one list $\mathcal{T}_\ell$ per value class V_ℓ.

For maintaining and updating the pointer κ_t, we generate a list $\mathcal{C}_t$ of all configurations $c \in \mathcal{C}'$ with $n_{c,t} \geq 1$. By iterating through each $c \in \mathcal{C}'$, we can add c to the list of t if $n_{c,t} \geq 1$. We additionally store $n_{c,t}$ and α_c in the list $\mathcal{C}_t$. While iterating through the configurations, we additionally compute $\bar{n}_t = \sum_{c \in \mathcal{C}'} \bar{y}_c n_{c,t}$ and store $\bar{n}_t$ in the same list as the item types $\mathcal{T}_\ell$. Note that since the list of $\mathcal{C}'$ by definition is ordered by indices, the created lists $\mathcal{C}_t$ are also sorted by indices. For each item type, we point κ_t to the first knapsack of the first added configuration c and set $\eta_t = n_{c,t}$. If the list of an item type remains empty, we set $\kappa_t = 0$. Since each configuration contains at most $\frac{1}{\varepsilon}$ item types, the lists $\mathcal{C}_t$ can be generated in time $\mathcal{O}(\frac{|\mathcal{C}'|}{\varepsilon})$.

Now consider a queried big item j. In time $\mathcal{O}(\log n)$ we can decide whether j has already been queried in the current round. If not, let V_ℓ be the value class of j, which was computed upon arrival of j. If $\ell < \bar{\ell}$, j does not belong to the current solution and no data structures need to be updated. Otherwise, the type of j is determined by accessing the item types $\mathcal{T}_\ell$ in time $\mathcal{O}(\log \frac{\log n}{\varepsilon})$. Once t is determined, $\bar{n}_t$ can be added to the left boundary of type t in order to determine if j is packed or not. If j belongs to the current solution, pointer κ_t dictates the answer to the query.

In order to update κ_t and η_t, we extract c, the current configuration of knapsack κ_t in time $\mathcal{O}(\log |\mathcal{C}'|)$ by binary search over the list α. If $\kappa_t + 1 < \alpha_{c+1}$, κ_t is increased by one and $\eta_t = n_{c,t}$ in constant time. If not, the next configuration c' containing t can be found with binary search over the list $\mathcal{C}_t$ in time $\mathcal{O}(\log |\mathcal{C}'|)$. If no such configuration is found, we set $\kappa_t = 0$. Otherwise, we set $\kappa_t = \alpha_{c'}$ and $\eta_t = n_{c,t}$. Overall, queries for big items can be answered in time $\mathcal{O}\big(\max\{\log |\mathcal{C}'|, \log \frac{\log n}{\varepsilon}\}\big)$. Observing that $|\mathcal{C}'| \in \mathcal{O}(\mathcal{T}) = \mathcal{O}\big(\frac{\log^2 n}{\varepsilon^4}\big)$ completes the proof. □

Lemma D.4. *The above mentioned data structures for small items can be generated in time $\mathcal{O}(\frac{\log^2 n}{\varepsilon^5})$. Queries for small items can be answered in time $\mathcal{O}\big(\max\big\{\log \frac{\log n}{\varepsilon}, \frac{1}{\varepsilon}\big\}\big)$.*

Proof. We initialize $\kappa^r = 1$ and $\rho = S - s_1$ where s_1 is the total size of the configuration assigned to the first knapsack. For packing cut items, we use the pointer κ^c to the current knapsack for "cut" items while η^f stores the remaining slots of small items. We initalize these values with $\kappa^c = (1-\varepsilon)m - \varepsilon m + 1$ and $\eta^c = \frac{1}{\varepsilon}$. These initializations can be computed in time $\mathcal{O}(\log |\mathcal{C}'|)$ (for accessing s_1) while the numbers are bounded by S and m.

Now consider a queried small item j. In time $\mathcal{O}(\log n)$ we can decide whether j has already been queried in the current round. In constant time, we can decide whether $j > j^\star + 1$. If $j > j^\star$, the answer is NOT SELECTED. If $j = j^\star + 1$, we return m. Both answers can be determined in constant time with numbers bounded by m. If $j \leq j^\star$, the algorithm only needs to decide if j is packed into κ^r or κ^c, which can be done in constant time. Finally, κ^r, κ^c as well as ρ and η^c need to be updated. While κ^c, κ^r and η^c can be updated in constant time, we need to compute the configuration c and remaining capacity $S - s_c$ of knapsack κ^r if the pointer is increased. By using binary search over the list α, the configuration can be determined in time $\mathcal{O}(\log |\mathcal{C}'|)$. Once the configuration is known, ρ can be calculated in time $\mathcal{O}\big(\frac{1}{\varepsilon}\big)$ with numbers bounded by S. Overall, queries for small items can be answered in time $\mathcal{O}\big(\max\big\{\log |\mathcal{C}'|, \frac{1}{\varepsilon}\big\}\big)$.

Using that $|\mathcal{C}'| \in \mathcal{O}(|\mathcal{T}|) = \mathcal{O}\big(\frac{\log^2 n}{\varepsilon^4}\big)$ concludes the proof. □

Lemma D.5. *A query for the solution value can be answered in time $\mathcal{O}\big(\frac{\log^3 n}{\varepsilon^4}\big)$.*

Proof. The value achieved by the small items, v_S can be computed with on prefix computation of the first $j^\star + 1$ items in the density-sorted tree for small items in time $\mathcal{O}(\log n)$ by Lemma 3.2.

For computing the value of a big item, we consider each value class V_ℓ with $\bar{\ell} \le \ell \le \ell_{\max}$ individually. There are at most $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$ many values classes by Lemma C.1. For one value class, in time $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$, iterate through the item types t. For each item type, we can access the total value of the first $\bar{n}_t$ items in time $\mathcal{O}(\log n)$ by Lemma 3.2.

Combining these two bounds gives the running time claimed in the lemma. □

Lemma D.6. *A query for the complete solution can be answered in time* $\mathcal{O}\big(|P|\frac{\log^3 n}{\varepsilon^3}\max\{\log|\mathcal{C}'|, \log\frac{\log n}{\varepsilon}\}\big)$ *where P is our solution.*

Proof. The small items belonging to a solution can be accessed in time $\mathcal{O}((j^\star + 1)\log n)$ by Lemma 3.2. Lemma D.4 then ensures that their knapsacks can be determined in time $\mathcal{O}\big(\max\big\{\log|\mathcal{C}'|, \frac{1}{\varepsilon}\big\}\big)$.

For big items, we consider again at most $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$ many value classes individually. In time $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$, we access the boundaries of the corresponding item types. In time $\mathcal{O}(\bar{n}_t \log n)$ we can access the $\bar{n}_t$ items of type t belonging to our solutions by Lemma 3.2. Lemma D.3 ensures that their knapsacks can be determined in time $\mathcal{O}\big(\max\{\log|\mathcal{C}'|, \log\frac{\log n}{\varepsilon}\}\big)$

In total, this bounds the running time by $\mathcal{O}\big(|P|\frac{\log^3 n}{\varepsilon^4}\max\{\log|\mathcal{C}'|, \log\frac{\log n}{\varepsilon}\}\big)$. □

E Knapsacks with Resource Augmentation

In this section, we consider instances for MULTIPLE KNAPSACK with many knapsacks and arbitrary capacities. We show how to efficiently maintain a $(1-\varepsilon)$-approximation when given $L = (\frac{\log n}{\varepsilon})^{O(\frac{1}{\varepsilon})}$ additional knapsacks that have the same capacity as a largest knapsack in the input instance. The algorithm will again solve the LP relaxation of a configuration ILP and round the obtained solution to an integral packing. However, in contrast to the problem for identical knapsacks, not every configuration fits into every knapsack and we therefore cannot just reserve a fraction of knapsacks in order to pack the rounded configurations since the knapsack capacities might not suffice. For this reason, we employ resource augmentation in the case of arbitrary knapsack capacities. While we may pack items into the additional knapsacks, an optimal solution is not allowed to use them. Again, we assume that item values are rounded to powers of $(1+\varepsilon)$ which results in value classes V_ℓ of items with value $v_j = (1+\varepsilon)^\ell$. We prove the following theorem.

Theorem 6.3. *There is a dynamic algorithm for* MULTIPLE KNAPSACK *that, when given* $L = (\frac{\log n}{\varepsilon})^{O(1/\varepsilon)}$ *additional knapsacks as resource augmentation, achieves an approximation factor* $(1-\varepsilon)$ *with update time* $(\log n)^{\mathcal{O}(1/\varepsilon)}(\log S_{\max}\log v_{\max})^{\mathcal{O}(1)}$ *where* $S_{\max} := \max\{S_i : i \in [m]\}$. *Item queries are answered in time* $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$ *and a solution P can be output in time* $\mathcal{O}\big(|P|\frac{\log^4 n}{\varepsilon^6}\big)$.

Overview. We use dynamic linear grouping as developed in Section 5.1 in order to reduce the number of different item types to $O(\frac{\log^2 n}{\varepsilon^4})$. Given the set of item types $\mathcal{T}$, we decide whether a given item type is small or big with respect to a certain knapsack. Recall that an item j is called small with respect to a knapsack with capacity S_i if $s_j < \varepsilon S_i$ and big otherwise.

Using the item types, we group knapsacks of similar capacity in a way such that within a group any given item type is either small for all knapsacks in the group or it is big for all knapsacks in the group. We denote by $\mathcal{G}$ the set of all such groups. As there are $O(\frac{\log^2 n}{\varepsilon^4})$ item types, we have at most $O(\frac{\log^2 n}{\varepsilon^4})$ groups.

Within one group, we give an explicit packing of the big items into slightly less knapsacks than belonging to the group by solving a configuration ILP. For packing the small items, we use Lemma C.5. That is, we greedily fill up knapsacks with small items and pack any "cut" small item into the knapsacks that were left empty by the configuration ILP. However, since items classify as big in one knapsack group and as small in another group, instead of guessing the size of small items per knapsack group, we incorporate them into the configuration LP by reserving sufficient space for the small items in each group.

Data structures In this section, we maintain three different types of data structures: one tree for storing every item j together with its size s_j, its value v_j, and its value class ℓ_j sorted by non-decreasing time of arrival. We additionally store the knapsacks sorted in non-increasing capacity in one separate tree as well. For each value class V_ℓ, we additionally maintain one balanced binary tree for sorting the items with $\ell_j = \ell$ in order of non-decreasing size.

Algorithm

1) **Linear grouping of big items:** Guess $\ell_{\max}$, the index of the highest value class that belongs to OPT and use dynamic linear grouping with $J' = J$ and $n' = n$ to obtain $\mathcal{T}$, the set of item types t with their multiplicities n_t.

2) **Knapsack Grouping:** Consider the knapsacks sorted increasingly by their capacity and determine for each item size for which knapsacks a corresponding item would be big or small. This yields a set $\mathcal{G}$ of $O(\frac{\log^2 n}{\varepsilon^4})$ many knapsack groups. Denote by $\mathcal{F}_g$ the set of all item types that are small with respect to group g, and by S_g the total capacity of all knapsacks in group g. Let m_g be the number of knapsacks in group g and let $\mathcal{G}^{(1/\varepsilon)}$ be the groups in $\mathcal{G}$ with $m_g \geq \frac{1}{\varepsilon}$. For each $g \in \mathcal{G}^{(1/\varepsilon)}$, define $S_{g,\varepsilon}$ as the total capacity of the smallest εm_g many knapsacks in g. Similar to the ILP for identical knapsacks, the ILP reserves some knapsacks to pack small "cut" items. We distinguish between $\mathcal{G}^{(1/\varepsilon)}$ and $\mathcal{G} \setminus \mathcal{G}^{(1/\varepsilon)}$ to restrict only large enough groups g, i.e, $g \in \mathcal{G}^{(1/\varepsilon)}$, to the $(1-\varepsilon)m_g$ most valuable knapsacks of g.

3) **Configurations:** For each group $g \in \mathcal{G}$, create all possible configurations consisting of at most $\frac{1}{\varepsilon}$ items which are big with respect to knapsacks in g. This amounts to $O((\frac{\log^2 n}{\varepsilon^4})^{1/\varepsilon})$ configurations per group. Order the configurations decreasingly by size and denote the set of such configurations by $\mathcal{C}_g = \{c_{g,1}, c_{g,2} \dots c_{g,k_g}\}$. Let $m_{g,\ell}$ be the total number of knapsacks in group g in which we could possibly place configuration $c_{g,\ell}$. Further, denote by $n_{c,t}$ the number of items of type t in configuration c, and by s_c and v_c the size and value of c respectively.

4) **Configuration ILP:** Solve the following configuration ILP with variables y_c and $z_{g,t}$. Here, y_c counts how often a certain configuration c is used, and $z_{g,t}$ counts how many items of type t are packed in knapsacks of group g if type t is small with respect to g. Note that by the above definition of $\mathcal{C}_g$, we

may have duplicates of the same configuration for several groups.

$$\begin{array}{llll}
\max & \sum_{g\in\mathcal{G}}\sum_{c\in\mathcal{C}_g} y_c v_c + \sum_{g\in\mathcal{G}}\sum_{t\in\mathcal{F}_g} z_{g,t} v_t & & \\
\text{s.t.} & \sum_{h=1}^{\ell} y_{c_{g,h}} \leq m_{g,\ell} & \text{for all } g\in\mathcal{G}, \ell\in[k_g] \\
& \sum_{c\in\mathcal{C}_g} y_c \leq (1-\varepsilon)m_g & \text{for all } g\in\mathcal{G}^{(1/\varepsilon)} \\
& \sum_{c\in\mathcal{C}_g} y_c s_{c_{g,h}} + \sum_{t\in\mathcal{F}_g} z_{g,t}s_t \leq S_g & \text{for all } g\in\mathcal{G}\setminus\mathcal{G}^{(1/\varepsilon)} \\
& \sum_{c\in\mathcal{C}_g} y_c s_{c_{g,h}} + \sum_{t\in\mathcal{F}_g} z_{g,t}s_t \leq S_g - S_{g,\varepsilon} & \text{for all } g\in\mathcal{G}^{(1/\varepsilon)} \\
& \sum_{g\in\mathcal{G}}\sum_{c\in\mathcal{C}_g} y_c n_{c,t} + \sum_{g\in\mathcal{G}:t\in\mathcal{F}_g} z_{g,t} \leq n_t & \text{for all } t\in\mathcal{T} \\
& y_c \in \mathbb{Z}_{\geq 0} & \text{for all } g\in\mathcal{G}, c\in\mathcal{C}_g \\
& z_{g,t} \in \mathbb{Z}_{\geq 0} & \text{for all } t\in\mathcal{T}, g\in\mathcal{G} \\
& z_{g,t} = 0 & \text{for all} t\in\mathcal{T}, g\in\mathcal{G} \,:\, t\notin\mathcal{F}_g
\end{array} \tag{P}$$

The first inequality ensures that the configurations chosen by the ILP actually fit into the knapsacks of the respective group while the second inequality ensures that an ε-fraction of knapsacks in $\mathcal{G}_{1/\varepsilon}$ remains empty for packing small "cut" items. The third and fourth inequality guarantee that the total volume of large and small items together fits within the designated total capacity of each group. Finally, the fifth inequality makes sure that only available items are used by the ILP.

5) **Obtaining an integral solution:** After relaxing the above ILP and allowing fractional solutions, we are able to solve it efficiently. Let OPT_{LP} be an optimal (fractional) solution to (P) with objective function value v_{LP}. With Lemma C.4 we obtain an integral solution that uses the additional knapsacks given by the resource augmentation with value at least v_{LP}. Let P_F denote this final solution.

6) **Packing small items:** Observe that small item types $t \in \mathcal{F}_g$ are only packed fractionally by P_F. Lemma C.5 provides us with a way to pack the small items integrally.

Analysis We start again by showing that the loss in the objective function value due to the linear grouping of items is bounded by a facor of at most $\frac{(1-\varepsilon)(1-2\varepsilon)}{(1+\varepsilon)^2}$. To this end, let OPT be an optimal solution to the current, non-rounded instance and let J be the set of items with values already rounded to powers of $(1+\varepsilon)$. By setting $J' = J$, we apply Lemma 3.1 and Lemmas C.1 to C.3 to obtain the following corollary. Here, $\text{OPT}_{\mathcal{T}}$ is the optimal solution for the instance specified by the item types $\mathcal{T}$ with multiplicities n_t.

Corollary E.1. *Let* OPT *and* $\text{OPT}_{\mathcal{T}}$ *be defined as above. Then,* $v(\text{OPT}_{\mathcal{T}}) \geq \frac{(1-\varepsilon)(1-2\varepsilon)}{1+\varepsilon} v(\text{OPT})$.

We have thus justified the restriction to item types in $\mathcal{T}$ instead of packing the actual items. In the next two lemmas, we show that (P) is a linear programming formulation of the DYNAMIC MULTIPLE KNAPSACK problem on the item set $\mathcal{T}$ and that we can obtain a feasible integral packing (using resource augmentation) if we have a fractional solution (without resource augmentation) to (P). Let v_{LP} be the optimal objective function value of the LP relaxation of (P).

Similar to the proof of Lemma 5.7 we restrict a given optimal solution $\text{OPT}_{\mathcal{T}}$ to the $(1-2\varepsilon)m_g$ most valuable knapsacks if $m_g \geq \frac{1}{\varepsilon}$ and otherwise we do not restrict the knapsacks at all.

Lemma E.2. *It holds that* $v_{LP} \geq (1-2\varepsilon)v(\text{OPT}_{\mathcal{T}})$.

Proof. We show the statement by explicitly stating a solution (y, z) that is feasible for (P) and achieves an objective function value of at least $(1-2\varepsilon)v(\text{OPT}_{\mathcal{T}})$.

Consider a feasible optimal packing $\text{OPT}_{\mathcal{T}}$ for item types. The construction of (y, z) considers each group $g \in \mathcal{G}$. If $g \notin \mathcal{G}^{(1/\varepsilon)}$, let y_c count how often a configuration $c \in \mathcal{C}_g$ is used in $\text{OPT}_{\mathcal{T}}$. Moreover, let $z_{g,t}$ denote how often an item that is small with respect to g is used in $\text{OPT}_{\mathcal{T}}$. By construction, the first and the third constraint of (P) are satisfied. The solution (y, z) restricted to group g achieves the same solution value as $\text{OPT}_{\mathcal{T}}$ restricted to the same knapsacks.

If $g \in \mathcal{G}^{(1/\varepsilon)}$, i.e., if there are at least $\frac{1}{\varepsilon}$ knapsacks in group g, consider the $\lfloor(1-\varepsilon)m_g\rfloor$ most valuable knapsacks in group g packed by $\text{OPT}_{\mathcal{T}}$. Define y_c to count how often $\text{OPT}_{\mathcal{T}}$ uses configuration $c \in \mathcal{C}_c$ in this reduced knapsack set and let $z_{g,t}$ denote how often $\text{OPT}_{\mathcal{T}}$ uses item type $t \in \mathcal{F}_g$ in these knapsacks. Clearly, this solution satisfies the first constraint of (P). By construction, $\sum_{c\in\mathcal{C}_g} y_c \le \lfloor(1-\varepsilon)m_g\rfloor \le (1-\varepsilon)m_g$ and, hence, the second constraint of the ILP is also satisfied. Cleary, the $\lfloor(1-\varepsilon)m_g\rfloor$ most valuable knapsacks can be packed into the $\lfloor(1-\varepsilon)m_g\rfloor$ largest knapsacks in g, which implies the feasibility for the fourth constraint of the ILP. Observe that $\lfloor(1-\varepsilon)m_g\rfloor \ge (1-\varepsilon)m_g - 1 \ge (1-2\varepsilon)m_g$. Thus, the value of the corresponding packing is at least a $(1-2\varepsilon)$ fraction of the value that $\text{OPT}_{\mathcal{T}}$ obtains with group g.

As (y, z) uses no more items of a certain item type than $\text{OPT}_{\mathcal{T}}$ does, the last constraint of the ILP is also satisfied. Hence, (y, z) is feasible and

$$v_{\text{LP}} \ge \sum_{g\in\mathcal{G}}\Big(\sum_{c\in\mathcal{C}_g} y_c v_c + \sum_{t\in\mathcal{F}_g} z_{g,t} v_t\Big) \ge (1-2\varepsilon)v(\text{OPT}_{\mathcal{T}}).$$

□

The next corollary shows how to round any fractional solution of (P) to an integral solution (possibly) using additional knapsacks given by resource augmentation. It follows immediately from Lemma C.4.

Corollary E.3. *Any feasible solution (y, z) of the LP relaxation of (P) with objective function value v can be rounded to an integral solution using at most L extra knapsacks with total value at least v.*

In the next lemma, we bound the value obtained by our algorithm in terms of OPT, the optimal solution for a given input. Let P_F be the solution returned by our algorithm.

Lemma E.4. *Let P_F be defined as above. Then, $v(P_F) \ge \frac{(1-2\varepsilon)^2(1-\varepsilon)}{(1+\varepsilon)^2} v(\text{OPT})$.*

Proof. Observe that our algorithm outputs the solution P_F^* with the maximum value over all guesses of $\ell_{\max}$, the highest value class in OPT. Hence, we give a guess $\ell_{\max}$ and a corresponding solution P that satisfies $v(P) \ge \frac{(1-2\varepsilon)^2(1-\varepsilon)}{(1+\varepsilon)^2} v(\text{OPT})$.

Fix an optimal solution OPT and let $\ell_{\max} := \max\{l : V_\ell \cap \text{OPT} \neq \emptyset\}$ and set $\bar{\ell} := \ell_{\max} - \left\lceil \frac{\log(n/\varepsilon)}{\log(1+\varepsilon)} \right\rceil$. Then, $\ell_{\max}$ is considered in some round of the algorithm. Hence, let v_{ILP} be the optimal solution to the configuration ILP (P) and let v_{LP} be the solution value of its LP relaxation. Corollary E.3 provides a way to round the corresponding LP solution (y, z) to an integral solution $(\bar{y}, \bar{z})$ using at most L extra knapsacks with objective function value at least $v_{\text{LP}} \ge v_{\text{ILP}}$. The construction of $(\bar{y}, \bar{z})$ guarantees that only small items in the original knapsacks might be packed fractionally.

Consider one particular group g. Lemma C.5 shows how to pack the fractional small items selected by $(\bar{z}_g)$ into εm_g extra knapsacks. If $m_g < \frac{1}{\varepsilon}$, we use one extra knapsack per group to store the fractional items. If $m_g \ge \frac{1}{\varepsilon}$, $g \in \mathcal{G}^{(1/\varepsilon)}$ which implies that the configuration ILP (and its relaxation) already reserved $\lceil \varepsilon m_g \rceil$ knapsacks of this group for packing small items. Hence, P is feasible. By Corollary E.1 and Lemma E.2 we have that

$$v(P_F) \ge v(P) \ge \frac{(1-2\varepsilon)^2(1-\varepsilon)}{(1+\varepsilon)^2} v(\text{OPT}).$$

□

Now, we bound the running time of our algorithm.

Lemma E.5. *The update time of the dynamic algorithm is bounded by* $\left(\frac{1}{\varepsilon}\log n\right)^{\mathcal{O}(1/\varepsilon)}(\log m \log S_{\max} \log v_{\max})^{\mathcal{O}(1)}$.

Proof. By assumption, upon arrival, the value of each item is rounded to natural powers of $(1+\varepsilon)$. The algorithm starts with guessing $\ell_{\max}$ the highest value class to be considered in the current iteration. There are $\log v_{\max}$ many guesses possible where $v_{\max}$ is the highest value appearing in the current instance.

By Lemma 5.4, the harmonic rounding of all items has at most $\mathcal{O}\left(\frac{\log^4 n}{\varepsilon^4}\right)$ iterations. The size of the appearing numbers is bounded by $\mathcal{O}(\log\max\{S_{\max},\overline{v}\})$, where $S_{\max} := \max S_i$ is the maximal capacity of the given knapsacks.

Let the knapsacks be sorted by increasing capacity and stored in a binary balanced search tree as defined in Lemma 3.2. Then, the index of the smallest knapsack i with $S_i \geq S$ or the largest knapsack with $S_i \leq S$ can be determined in time $\mathcal{O}(\log m)$, where S is a given number. Thus, the knapsack groups depending on the item types can be determined in time $\mathcal{O}(\log m \frac{\log^2 n}{\varepsilon^4})$ as the number of item types is bounded by $\mathcal{O}(\frac{\log^2 n}{\varepsilon^4})$. The number of big items per knapsack is bounded by $\frac{1}{\varepsilon}$ and, hence, the number of configurations is bounded by $\mathcal{O}\Big(\frac{\log^2 n}{\varepsilon^4}\big(\frac{\log^2 n}{\varepsilon^4}\big)^{\frac{1}{\varepsilon}}\Big)$.

The number of variables in the considered configuration ILP is bounded by $N = \mathcal{O}\Big(\frac{\log^2 n}{\varepsilon^4}\big(\big(\frac{\log^2 n}{\varepsilon^4}\big)^{\frac{1}{\varepsilon}} + \frac{\log^2 n}{\varepsilon^4}\big)\Big) = \mathcal{O}\big(\frac{\log^{3/\varepsilon} n}{\varepsilon^{5/\varepsilon}}\big)$. Hence, there is a polynomial function $g(N, \log S_{\max}, \log v_{\max})$ that bounds the running time of finding an optimal solution to the LP relaxation of the configuration ILP. Clearly, setting up and rounding the fractional solution is dominated by solving the LP.

Combining everything, we can bound the running time of the algorithm by $\left(\frac{1}{\varepsilon}\log n\right)^{\mathcal{O}(1/\varepsilon)}(\log m \log S_{\max} \log v_{\max})^{\mathcal{O}(1)}$.

In similar time, we can store y and z, the obtained solutions to the configuration LP. Let y' and z' be the variables obtained by (possibly) rounding down y and z and let y'' and z'' be the variables assigned to the resource augmentation by Lemma C.4. Clearly, obtaining these variables is dominated by solving the LP relaxation of the configuration ILP. □

Answering Queries Since we only store implicit solutions, it remains to show how to answer the corresponding queries. In order to determine the relevant parameters of a particular item, we assume that all items are stored in one balanced binary search tree that allows us to access one item in time $\mathcal{O}(\log n)$ by Lemma 3.2. We additionally assume that this balanced binary search tree also stores the value class of an item. We use again the round parameter $t(j)$ and the corresponding knapsack $k(j)$ to cache given answers in order to stay consistent between two updates. If j was NOT SELECTED in round $t(j)$, we represent this by $k(j) = 0$. We assume that these two parameters are stored in the same binary search tree that also stores the items and, thus, can be accessed in time $\mathcal{O}(1)$.

We now design an algorithm for non-cached items. The high-level idea is identical to the algorithm developed in Section 5.2. As the knapsacks have different capacities in this section, the size class of an item depends on the particular knapsack group, i.e., an item can be big with respect to one knapsack and small with respect to another. Thus, the distinction between small and big items is not possible anymore and needs to be handled carefully. Table 1 gives an overview over the parameters and counters used to answer queries between two updates.

We assume that the knapsacks are sorted by decreasing capacity and stored in one binary search tree together with S_i, the capacity of the knapsacks. The knapsacks given by the resource augmentation are stored in three different lists $R^{(y)}$, $R^{(z)}$, and $R^{(\varepsilon)}$ indicating whether they are needed due to rounding y or z, or because $m_g < \frac{1}{\varepsilon}$. The knapsack groups are stored in the list $\mathcal{G}$ sorted by decreasing capacity. For each group, we additionally store m_g, the number of knapsacks in group g.

Let y', y'', z', and z'' be the implicit solution of the algorithm. Here $*'$ refers to packing configurations or items into the original knapsacks while $*''$ refers to the knapsacks given by resource augmentation. Let $\mathcal{C}'_g$ be the set of configurations c with $y'_{c,g} + y''_{c,g} \geq 1$ ordered in decreasing size s_c and stored in one list per group. In the following, we use the position of a configuration $c \in \mathcal{C}'_g$ in that list as the index of c. For mapping the configurations to knapsacks we assign the knapsacks $\sum_{g'=1}^{g-1} m_{g'} + 1, \dots, \sum_{g'=1}^{g} m_{g'-1} + \sum_{c'=1}^{c} y_c$ to configuration c.

For each item type t, we maintain a pointer γ_t to the group where the next queried item of type t is supposed to go. If t is big with respect to γ_t, we use again the pointer κ_t to refer to the particular knapsack where the next item of type t goes while η_t stores how many slots κ_t still has for items of type t. Because of resource augmentation, κ_t may point to a knapsack in $R^{(y)}$, the additional knapsacks for rounding y.

If t is small with respect to γ_t, we use group pointers $\kappa^r_{\gamma_t}$ and $\kappa^c_{\gamma_t}$ to refer to the knapsack for packing items regularly or to the position for packing cut items. If $m_g < \frac{1}{\varepsilon}$, than $R_\varepsilon(g)$ is used for packing cut items. As the number of items of type t assigned to group g as small items is determined by $z'_{g,t} + z''_{g,t}$, we additionally use the counter η_t, initialized with $z'_{\gamma_t,t} + z''_{\gamma_t,t}$, to reflect how many slots group γ_t still has for items of type t. As before, ρ_g refers to the remaining space for small items in the knapsack κ^r_g. Because of resource augmentation, both knapsack pointers may point to a knapsack given by resource augmentation.

Answering Item Queries.

1) **Check cache.** Let τ be the current round and let j be the queried item. If $t(j) = \tau$, return $k(j)$.

2) **Answer queries for non-cached items.** Set $t(j) = \tau$ and determine t, the type of j. Let γ be the group of t. If $\gamma = 0$, return NOT SELECTED. Decide if j is small or big with respect to the group γ.

   Small items. If $\eta_t = z''_{\gamma,t}$, determine if j goes to the resource augmentation $R^{(z)}$:

   If $z''_{\gamma,t} = 1$, set $k(j)$ to the knapsack in $R^{(z)}$ reserved for $z''_{\gamma,t}$ and increase γ_t to the next group for type t. If no such group exists, set $\gamma_t = 0$. Otherwise, update η_t and possibly κ_t accordingly.

   If $z''_{\gamma,t} = 0$, increase γ_t to the next group for type t and go to Step 2. If no such group exists, set $\gamma_t = 0$, $k(j) = 0$, and return NOT SELECTED.

   Otherwise, determine if j is packed regularly or as a cut item. If $s_t \leq \rho_\gamma$, return $k(j) = \kappa^r_\gamma$ and decrease ρ_γ accordingly. Otherwise, return $k(j) = \kappa^c_\gamma$ and increase κ^c_γ to the next position for "cut" items in group γ.

   Big items. If $\gamma_t = 0$, return NOT SELECTED and set $k(j) = 0$. Otherwise, return $k(j) = \kappa_t$ and decrease η_t by one. If this implies $\eta_t = 0$, let c be the configuration of κ_t.

   If $\kappa_t \in R^{(y)}$, let c' be the next configuration for type t in group γ and update κ_t and η_t accordingly. If no such configuration exists, increase γ_t to the next group for type t and update κ_t and η_t accordingly. If no such group exists, set $\gamma_t = 0$.

   If κ_t belongs to the original knapsacks and is the last knapsack assigned to configuration c, check if there is resource augmentation for configuration c. In this case, point κ_t to the knapsack reserved for rounding $y''_{c,\gamma}$. Otherwise, let c' be the next configuration for type t in group γ and update κ_t and η_t accordingly. If no such configuration exists, increase γ_t to the next group for type t and update κ_t and η_t accordingly. If no such group exists, set $\gamma_t = 0$.

   Otherwise, increase κ_t by one and update η_t accordingly.

Table 1: Counters and parameters used during querying items.

Counter/Pointer	Meaning
$\mathcal{C}'_g$	Configurations that are used by group g
$\alpha_{c,g}$	First knapsack with configuration c in group g
$R^{(y)}_{c,g}$	Knapsack in R_y used for group g and configuration c
$R^{(z)}_{g,t}$	Knapsack in R_z used for group g and type t
$\mathcal{G}_t$	Knapsack groups where items of type t are packed
γ_t	Current knapsack group where items of type t are packed
κ^r_g	Current knapsack in g for packing small items regularly
κ^c_g	Current knapsack in g (or in R_ε) for packing cut small items
ρ^r_g	Remaining capacity in κ^r_g for packing small items
η^r_f	Remaining number of slots for small items in κ^r_f
$\mathcal{C}_{g,t}$	List of configurations $c \in \mathcal{C}'_g$ with $n_{c,t} \geq 1$
κ_t	Current knapsack for packing items of a big type t
η_t	Remaining number of slots for items of type t in κ_t or in γ_t depending on the size of t with respect to γ_t.

Answering the Solution Value Query.

1) **Value per item type.** For each item type t, calculate v_t, the total value of the first $\bar{n}_t$ items with prefix computation.
2) **Value.** Return $\sum_{t\in\mathcal{T}} v_t$.

Answering the Solution Query.

- For each item type t, query the first $\bar{n}_t$ items and return these items with their knapsacks.

Lemma E.6. *The query algorithms return a feasible and consistent solution obtaining the total value given by the implicit solution.*

Proof. By construction of $k(j)$ and $t(j)$, the solution returned by the query algorithm is consistent between updates.

Observe that y' and z' is a feasible solution to the configuration ILP (P). Hence, showing that the algorithm does not assign more than $y'_{c,g}$ times configuration c and not more than $z'_{g,t}$ items of type t to group g is sufficient for having a feasible packing of the corresponding elements into the $\lfloor(1-\varepsilon)m_g\rfloor$ largest knapsacks if $m_g \geq \frac{1}{\varepsilon}$ or into the m_g knapsacks of group g if $m_g < \frac{1}{\varepsilon}$.

If the item type t is small with respect to the group g, then at most $z'_{g,t}$ items of type t are packed in group g. Then, Lemma C.5 ensures that all small items assigned to group g fit in the regular and the "cut"-items knapsack. Moreover, the treatment of $\eta_t = z''_{g,t}$ guarantees that the value obtained by small items packed in g and its additional knapsacks is as claimed by the implicit solution.

If t is big with respect to group g, then the construction of κ_t and η_t ensure that exactly $\sum_{c\in\mathcal{C}'_g}(y'_c + y''_c)n_{c,t}$ items of type t are packed in group g and in $R^{(y)}$. Hence, the total value achieved is as given by the implicit solution. □

The next lemmas are concerned with the running time of the algorithms used for answering queries. Table 1 summarizes the counters and pointers used in the proof.

Lemma E.7. *The above mentioned data structures can be generated in $\mathcal{O}\left(\frac{\log^3 n}{\varepsilon^5}\frac{\log^{2/\varepsilon} n}{\varepsilon^{4/\varepsilon}}\right)$ many iterations. Queries for a particular item can be answered in $\mathcal{O}\left(\frac{\log n}{\varepsilon^2}\right)$ many steps.*

Proof. We start by retracing the steps of the dynamic linear grouping in order to obtain the set $\mathcal{T}$ of item types. We store the types $\mathcal{T}_\ell$ of one value class in one list, sorted by non-decreasing size. By Lemma 5.4, the set $\mathcal{T}$ can be determined in time $\mathcal{O}(\frac{\log^4 n}{\varepsilon^4})$.

We first argue about the generation of the data structures and the initialization of the various pointers and counters. We start again by generating a list α_g for each group g where $\alpha_{c,g}$ stores the first (original) knapsack of configuration $c \in \mathcal{C}'_g$. Then, $\alpha_{c,g} = \alpha_{c-1,g} + y'_{c-1,g} + 1$ where $\alpha_{0,g} = 0$. Then, we set $R^{(z)}_{g,t} = \sum_{g'=1}^{g-1} \sum_{t \in \mathcal{T}} z''_{g,t}$ and $R^{(y)}_{c,g} = \sum_{g'=1}^{g-1} \sum_{t \in \mathcal{C}'_g} y''_{g,t}$, where $R^{(z)}_{g,t}$ corresponds to the resource augmentation needed because of rounding $z_{g,t}$ and $R^{(y)}_{c,g}$ corresponds to the resource augmentation caused by rounding $y_{c,g}$. Then, these lists can be generated by iterating through the list $\mathcal{C}'_g$ for each group g in time $\mathcal{O}(\sum_{g \in \mathcal{G}} |\mathcal{C}_g|) = \mathcal{O}(\frac{\log^2 n}{\varepsilon^4} \frac{\log^{2/\varepsilon} n}{\varepsilon^{\varepsilon/4}})$.

For maintaining and updating the pointer γ_t, we generate the list $\mathcal{G}_t$ that contains all groups g where items of type t are packed in the implicit solution. By iterating through the groups once more and checking $\sum_{c \in \mathcal{C}'_g} (y'_{c,g} + y''_{c,g}) n_{t,c} \geq 1$ or $z'_{t,g} + z''_{g,t} \geq 1$, we can add the corresponding groups g to $\mathcal{G}_t$. Then, γ_t points to the head of the list. Note that the appearing numbers are bounded by $\mathcal{O}(|\mathcal{C}_g| n)$. While iterating through the groups, we also calculate $\bar{n}_t = \sum_{g \in \mathcal{G}} \left(\sum_{c \in \mathcal{C}'_g} (y'_{c,g} + y''_{c,g}) + z'_{t,g} + z''_{t,g} \right)$ and store the corresponding value together with the item type. The lists $\mathcal{G}_t$ can be generated in $\mathcal{O}(|\mathcal{T}| \sum_{g \in \mathcal{G}} |\mathcal{C}_g|) = \mathcal{O}\left(\frac{\log^3 n}{\varepsilon^5} \frac{\log^{2/\varepsilon} n}{\varepsilon^{4/\varepsilon}} \right)$ many iterations.

For maintaining and updating the pointer κ_t we create the list $\mathcal{C}_{g,t}$ storing all configurations $c \in \mathcal{C}'_g$ where t is packed as big item. While iterating through the groups and creating $\mathcal{G}_t$, also checking $y'_{c,g} + y''_{c,g} \geq 1$ for each configuration allows us to add c to the list $\mathcal{C}_{g,t}$ where we also store $n_{c,t}$. Initially, we point κ_t to the head of the first group g where t is packed as big item. If c is the corresponding configuration, we start with $\eta_t = n_{c,t}$. Then, the time needed for this is bounded by $\mathcal{O}(|\mathcal{T}| \sum_{g \in \mathcal{G}} |\mathcal{C}_g|) = \mathcal{O}\left(\frac{\log^3 n}{\varepsilon^5} \frac{\log^{2/\varepsilon} n}{\varepsilon^{4/\varepsilon}} \right)$.

The pointer κ^r_g is initialized with $\kappa^r_g = \sum_{g'=1}^{g-1} m_g + 1$. By using binary search on the list $\mathcal{C}'_g$, we get s_1, the total size of configuration 1 assigned to κ^r_g, and binary search over the knapsacks allows us to obtain $S_{\kappa^r_g}$, the capacity of knapsack κ^r_g. Then, $\rho_g = S_{\kappa^r_g} - s_1$ can be initialized in time $\mathcal{O}(\sum_{g \in \mathcal{G}} (\log(|\mathcal{C}'_g|) + \log m) = \mathcal{O}\left(\frac{\log^2 n}{\varepsilon^5} \left(\log \frac{\log n}{\varepsilon} + \log m \right) \right)$.

If $m_g \geq \frac{1}{\varepsilon}$, we set $\kappa^c_g = \sum_{g'1=1}^{g-1} m_g + \lfloor (1-\varepsilon) m_g \rfloor + 1$ while $m_g < \frac{1}{\varepsilon}$ implies that κ^c_g points to the resource augmentation $R^{(\varepsilon)}$, i.e., $\kappa^c_g = R^{(\varepsilon)}_g = |\{g' \leq g : m_{g'} < \frac{1}{\varepsilon}\}|$. The time needed for initializing is $\mathcal{O}(|\mathcal{C}'_g|)$ while the numbers are bounded by m and $|\mathcal{C}'_g|$. In order to determine the position of the next cut item, we also maintain η_g, initialized with $\eta^c_g = \frac{1}{\varepsilon}$, that counts how many slots are still left in knapsack κ^c_g.

Now consider the query for an item j. In time $\mathcal{O}(\log n)$ we can decide if j has already been queried in the current round. Upon arrival of j, we calculated its value class V_ℓ. By retracing the steps of the linear grouping, the boundaries of value class V_ℓ can then be determined in time $\mathcal{O}(\frac{\log n}{\varepsilon^2})$. By binary search, the item type of j can then be determined in time $\mathcal{O}(\log \frac{\log n}{\varepsilon})$. Once the item type is determined, we check if j belongs to the first $\bar{n}_t$ items of this type. If not, then NOT SELECTED is returned. Otherwise, the pointer γ_t answers the question in which group item j is packed. The pointer γ_t is updated at most once *before* determining $k(j)$. Hence, the case distinction on the relative size of type t is invoked at most twice.

If j is small, the knapsack $k(j)$ can be determined in constant time by nested case distinction and having the correct pointer (either $\kappa^r_{\gamma_t}$ or $\kappa^c_{\gamma_t}$) dictate the answer. Returning the answer then takes time $\mathcal{O}(\log m)$. In order to bound the update time of the data structures, it suffices to consider the case where j is packed as a cut item since this implies the most updates. The capacity of the new knapsack $\kappa^r_{\gamma_t}$ can be determined in $\mathcal{O}(\log m)$ by binary search over the knapsack list while the configuration c of the new knapsack $\kappa^r_{\gamma_t}$ and its total size are determined by binary search over the list α_{γ_t} in time $\mathcal{O}\left(\log |\mathcal{C}'_{\gamma_t}| \right) = \mathcal{O}\left(\frac{\log(\log(n)/\varepsilon)}{\varepsilon} \right)$.

Then, $\rho_{\gamma_t} = S_{\kappa^r_{\gamma_t}} - s_c$ can be computed with constantly many operations while the appearing numbers are bounded by $\mathcal{O}(S_{\max})$. If $\eta^c_{\gamma_t} = 0$ after packing j in $\kappa^c_{\gamma_t}$, we increase the knapsack pointer by one and update $\eta^c_{\gamma_t} = \frac{1}{\varepsilon}$.

If j is big, the pointer κ_{γ_t} dictates the answer which can be returned in time $\mathcal{O}(\log m)$. For bounding the running time of the possibly necessary update operations, observe that η_t is updated in constant time with values bounded by n. If $\eta_t = 0$ after the update, the knapsack pointer κ_t needs to be updated as well. The most time consuming update operations are finding a new configuration c' and possibly even a new group g'. Finding $c' \in \mathcal{C}_{\gamma_t,t}$ can be done by binary search through the list $\mathcal{C}_{\gamma_t,t}$ in time $\mathcal{O}\left(\log|\mathcal{C}'_{\gamma_t,t}|\right) = \mathcal{O}\left(\frac{\log(\log(n)/\varepsilon)}{\varepsilon}\right)$. To update κ_t and η_t, we extract the configuration c' from the list α_{γ_t} and $n_{c,t}$ from the list $\mathcal{C}'_{\gamma_t,t}$ in $\mathcal{O}\left(\log|\mathcal{C}'_{\gamma_t}|\right) = \mathcal{O}\left(\frac{\log(\log(n)/\varepsilon)}{\varepsilon}\right)$ by binary search with values bounded by m and n respectively. If the algorithm needs to update γ_t as well, this can be done by binary search on the list $\mathcal{G}_t$ in time $\mathcal{O}(\log|\mathcal{G}_t|) = \mathcal{O}\left(\log\left(\frac{\log(n)}{\varepsilon}\right)\right)$.

In both cases, the running time of answering the query and possibly updating data structures is bounded by the running time of the linear grouping step, i.e., by $\mathcal{O}(\frac{\log n}{\varepsilon^2})$. □

Lemma E.8. *A query for the solution value can be answered in time $\mathcal{O}(1)$.*

Proof. For calculating the value of the current solution, we need to calculate v_t, the total value of the first $\bar{n}_t$ items. We do this by iterating through the value classes once and per value class, we iterate once through the list $\mathcal{T}_\ell$ to access the number $\bar{n}_t$. Then, we use prefix computation twice in order to access the total value of the first $\bar{n}_t$ items of type t. Lemma 3.2 bounds this time by $\mathcal{O}(\log n)$. By Lemma 5.3, the number of item types is bounded by $\mathcal{O}\big(\frac{\log^2 n}{\varepsilon^4}\big)$. Combining these two values bounds the total running time by $\mathcal{O}\big(\frac{\log^3 n}{\varepsilon^4}\big)$. As this time is clearly dominated by obtaining the implicit solution in the first place, we precalculate the solution value when computing the implicit solution value and store it to return it in $\mathcal{O}(1)$. □

Lemma E.9. *A query for the complete solution can be answered in time $\mathcal{O}\big(|P|\frac{\log^4 n}{\varepsilon^6}\big)$ where P is the current solution.*

Proof. For returning the complete solution, we iterate once through the value classes and for each value class, we iterate through the list $\mathcal{T}_\ell$ to access the number $\bar{n}_t$. Then, we use prefix computation on index for accessing the corresponding $\bar{n}_t$ items of type t. We access and query each item individually. Lemma E.7 bounds the running time of these queries by $\mathcal{O}\big(\frac{\log n}{\varepsilon^2}\big)$ while Lemma 3.2 bounds the running time for accessing item j. Lemma 5.3 bounds the number of item types. In total, the running time is bounded by $\mathcal{O}\big(|P|\frac{\log^4}{\varepsilon^6}\big)$ where P is the current solution. □

Proof of main result

Proof of Theorem 6.3. In Lemma E.4, we bound the approximation ratio achieved by our algorithm. Lemma E.5 gives the desired bound on the update time. Lemmas E.7 to E.9 additionally bound the time needed for answering a query. □

F Multiple Knapsack

Analysis. We consider the iteration in which all the guesses, $V_{\ell_{\max}}$, k and S_O are correct. Let $\mathcal{P}_1$ be the set of solutions on the ordinary knapsacks (without the additional virtual knapsack) and the special knapsacks such that the total size of ordinary items placed in special knapsacks lies in the range $[S_O, (1+\varepsilon)S_O]$.

Denote by $\textsc{Opt}_1$ a solution of highest value in $\mathcal{P}_1$. Altering $\textsc{Opt}$ by deleting the extra knapsacks gives a solution in $\mathcal{P}_1$ of value at least $(1-\varepsilon)\cdot v(\textsc{Opt})$. This holds since for correct guesses the yellow knapsacks by definition contribute at most an ε-fraction to $\textsc{Opt}$. Further, the correctness of the guessed S_O implies that the altered $\textsc{Opt}$ is indeed a packing in $\mathcal{P}_1$.

Observation F.1. *For* $\textsc{Opt}_1$ *defined as above, we have* $v(\textsc{Opt}_1) \geq (1-\varepsilon)\cdot v(\textsc{Opt})$.

Lemma F.2. *Consider an optimal solution* $\textsc{Opt}_O$ *to the ordinary subproblem, i.e., exclude items in* J_E *but include the virtual knapsack. Then* $v(\textsc{Opt}_O) \geq v(\textsc{Opt}_{1,O}) - 2\varepsilon \cdot v(\textsc{Opt})$*, where we use the shorthand* $\textsc{Opt}_{1,O} := (\textsc{Opt}_1 \cap J_O) \setminus J_E$.

Proof. Consider the ordinary items in $\textsc{Opt}_1$ that are not in J_E. Leave items on ordinary knapsacks in their current position and place ordinary items on special knapsacks into the virtual ordinary knapsack. The latter is possible with the exception of possibly an ε-fraction of the items (with respect to size) due to S_O being rounded down. Deleting the least dense items until the remainder fits into the virtual knapsack causes a loss of at most an ε-fraction of the value of $\textsc{Opt}_1$ plus an additional ordinary item j_O. This item j_O contributes at most an ε-fraction to $\textsc{Opt}$ as its value is not larger than that of the least valuable element in J_E which has a value of less than $\varepsilon v(\textsc{Opt})$. $\square$

Lemma F.3. *Let* P_F *be the final solution the algorithm computes. Then* $v(P_F) \geq (1-7\varepsilon)v(\textsc{Opt})$.

Proof. Consider P_O, the solution of the ordinary subproblem returned by the algorithm of Appendix E (including virtual knapsack and resource augmentation). We know that $v(P_O) \geq (1-\varepsilon)\cdot v(\textsc{Opt}_O) \geq v(\textsc{Opt}_{1,O}) - 3\varepsilon \cdot v(\textsc{Opt})$ by Theorem 6.3 and Lemma F.2.

Let $\textsc{Opt}_S := \textsc{Opt}_1 \cap J_S$, and $P_2 := P_O \cup \textsc{Opt}_S \cup J_E$. Then, $\textsc{Opt}_1 = \textsc{Opt}_{1,O} \cup (\textsc{Opt}_1 \cap J_E) \cup (\textsc{Opt}_1 \cap J_S)$ implies

$$\begin{aligned} v(\textsc{Opt}_1) &= v(\textsc{Opt}_{1,O}) + v(\textsc{Opt}_1 \cap J_E) \;+\; v(\textsc{Opt}_1 \cap J_S) \\ &\leq v(P_O) + 3\varepsilon v(\textsc{Opt}) + v(J_E) + v(\textsc{Opt}_S) \\ &\leq v(P_2) + 3\varepsilon v(\textsc{Opt}). \end{aligned}$$

With Observation F.1 we then obtain $v(P_2) \geq v(\textsc{Opt}) - 4\varepsilon v(\textsc{Opt})$.

We now modify P_2 to obtain a solution P_3 that lacks the virtual ordinary knapsack and deals with bundles instead. Build $\frac{L_S}{\varepsilon}$ equal-sized bundles from P_O as in Step 5). Place these bundles fractionally on the remaining space of the special knapsacks that is left after $\textsc{Opt}_S$ is packed. This space is sufficient by definition of S_O and $\mathcal{P}_1$. Arrange the bundles such that the lowest-value ones are placed fractionally and removing them from the solution incurs a loss of at most $\varepsilon v(\textsc{Opt})$. Further, remove the items placed fractionally among bundles. Since there are at most $\frac{L_S}{\varepsilon}$ of these with value smaller than the $\frac{L_S}{\varepsilon^2}$ items in J_E, this incurs a loss of at most $\varepsilon v(\textsc{Opt})$.

Therefore, $v(P_3) \geq v(P_2) - 2\varepsilon v(\textsc{Opt})$. Moreover, the portion of P_3 on special knapsacks is a valid solution for the request sent to the special subproblem. Therefore, using Theorem 6.2, the overall solution P_F satisfies $v(P_F) \geq (1-7\varepsilon)v(\textsc{Opt})$. $\square$

Lemma F.4. *The algorithm has update time* $\left(\frac{1}{\varepsilon}\log(nv_{\max})\right)^{f(1/\varepsilon)} + \mathcal{O}(\frac{1}{\varepsilon}\log \overline{v} \log n)$*, where* f *is a quasi-linear function.*

Proof. Guessing k adds a factor of $\frac{1}{\varepsilon}$ to the update time. Placing the $\frac{L_S}{\varepsilon^2}$ most valuable ordinary items on extra knapsacks and removing them from data structures takes time $\mathcal{O}(\frac{L_S}{\varepsilon^2}\log n)$ which is within the time bound. The same holds for the updates of the ordinary and special data structures and for solving the subproblems with the algorithms of Appendices B and E.

Cutting the items placed in the virtual ordinary knapsack info $\frac{L_S}{\varepsilon}$ equal-sized bundles can be archived efficiently as follows. Compute the total size of these items, using the number of items used for each of the $\mathcal{O}(\frac{\log^2 n}{\varepsilon^4})$ item types and deduce the size of a bundle. Sort the item types, e.g., by value then size, and then iteratively pack items of the same type by computing how many items of this type fit in the next non-empty bundle. This takes time $\mathcal{O}(\frac{\log^2 n}{\varepsilon^4} \cdot \frac{L_S}{\varepsilon})$ which is sufficient.

Additionally, the maintenance of data structures is dominated in runtime by that of the subproblems. These takes time $\mathcal{O}(\frac{1}{\varepsilon} \log \overline{v} \log n)$ and cause the additive factor. □

G Proof of Theorem 3.3

G.1 Hardness of Approximation

The following theorems provides a justification why our algorithms for multiple knapsacks have different running times depending on the number of knapsacks. As Chekuri and Khanna [18] observed, MULTIPLE KNAPSACK with $m = 2$ does not admit an FPTAS unless P = NP.

Theorem G.1 (Proposition 2.1 in [18])**.** *If* MULTIPLE KNAPSACK *with two identical knapsacks has an FPTAS, then* PARTITION *can be solved in polynomial time. Hence there is no FPTAS for* MULTIPLE KNAPSACK *even with $m = 2$, unless* P = NP.

In the fully dynamic setting, this implies that there is no dynamic algorithm with running time polynomial in $\log n$ and $\frac{1}{\varepsilon}$ unless P = NP. There, we are able to extend this result to the case where $2n + m \leq \frac{1}{\varepsilon}$.

Theorem 3.3. *Unless* P = NP*, there is no fully dynamic algorithm for* MULTIPLE KNAPSACK *that maintains a $(1 - \varepsilon)$-approximate solution in update time polynomial in $\log n$ and $\frac{1}{\varepsilon}$, for $m < \frac{1}{3\varepsilon}$.*

Proof. Consider the strongly NP-hard problem 3-PARTITION where there are $3m$ items with sizes $a_j \in \mathbb{N}$ such that $\sum_{j=1}^{3m} a_j = mA$. The task is to decide whether there exists a partition $\bigcup_{i=1}^{m} J_i = [3m]$ such that $|J_i| = 3$ and $\sum_{j \in J_i} a_j = A$ for $1 \leq i \leq m$.

Consider the following instance for DYNAMIC MULTIPLE KNAPSACK: There are m knapsacks with $S = A$ and $3m$ many items. Each item corresponds to a 3-PARTITION item with $s_j = a_j$ and $v_j = 1$ for $1 \leq j \leq 3m$. Observe that the 3-PARTITION instance is a YES-instance if and only if the optimal solution to the KNAPSACK problem contains $3m$ items.

If DYNAMIC MULTIPLE KNAPSACK admits a dynamic algorithm with approximation guarantee at least $(1 - \varepsilon)$ and running time polynomial in $\frac{1}{\varepsilon}$ and $\log n_0$ where $m < \frac{1}{3\varepsilon}$, such an algorithm is able to optimally solve the KNAPSACK instance reduced from 3-PARTITION. Thus, such an algorithm decides 3-PARTITION in polynomial time which is not possible, unless P = NP. □

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