Dynamic Inverse Problems for Wave Phenomena

Thies Gerken

RTG 2224

Task

In this project we deal with second-order hyperbolic partial differential equations that include time- and space-dependent coefficients. In particular, we are interested in the inverse problems of identifying these coefficients based on their effect on the equation's solution.

As a simple example one can consider the identification of a function $q : [0, T] \times \Omega \to \mathbb{R}$ from the solution u of the wave equation

$$u'' - \Delta u + qu = f \text{ in } (0, T) \times \Omega$$

that is also assumed to possess homogeneous initial values $u(0) = u'(0) = 0$ and homogeneous Dirichlet boundary values.[1]

Possible applications of this kind of problems include the detection of moving objects with sound waves, new methods for nondestructive testing and motion compensation strategies, for example in the context of computerized tomography.

Abstract Inversion

To facilitate a wide range of possible equations, we generalized our analysis to the identification of time-dependent linear operators A, B and C from the solution u of the evolution equation

$$(C(t)u'(t))' + B(t)u'(t) + A(t)u(t) = f(t),$$

again with homogeneous initial conditions. This equation is stated in a Gelfand triple $V \subset H \subset V^*$, which may also be used to enforce boundary conditions. To be more precise, we assume $A(t) \in \mathcal{L}(V, V^*)$, whereas $B(t), C(t) \in \mathcal{L}(H)$.
This results in the forward operator

$$S\colon (A, B, C) \mapsto u,$$

and from an inverse problems perspective the following questions arise immediately:

- Is the forward operator S well-defined? Which spaces do we have to use?
- Is S Fréchet-differentiable? Most numerical methods also require the corresponding adjoints; do they have a simple characterization?
- Is the inverse problem ill-posed?

Conditions for A, B and C such that S is well-defined can be answered using standard techniques. However, for the differentiability of S the time-dependence of the unknowns causes problems, mainly because it is not straightforward to show higher regularity of the wave field u. However, after proving suitable regularity results[2], it turns out that all of the above questions can be answered positively.[3]

Application to the acoustic wave equation

One example for a parameter identification problem that can be solved using the abstract framework: The identification of a time- and space-dependent wave speed c and mass density ρ from the solution u of the wave equation

$$\frac{1}{\rho}\left(\frac{u'}{c^2}\right)' - \operatorname{div} \frac{\nabla u}{\rho} = f \quad \text{in } [0, T] \times \Omega$$

together with homogeneous initial- and boundary conditions. This fits into the abstract framework by setting $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ and

$$P(c, \rho) = (A_\rho, B_{c,\rho}, C_{c,\rho})$$

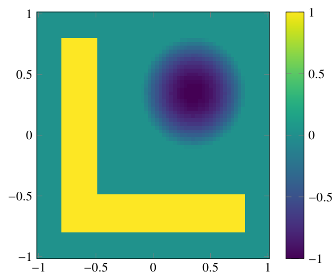
with $A_\rho(t)\phi = \operatorname{div} \frac{\nabla \phi}{\rho(t)}$, $B(t)\phi = \frac{\phi \rho'(t)}{\rho(t)^2 c(t)^2}$ and $C(t)\phi = \frac{\phi}{\rho(t)c(t)^2}$.
Furthermore, we can decompose the forward operator

$$F : (c, \rho) \mapsto u$$

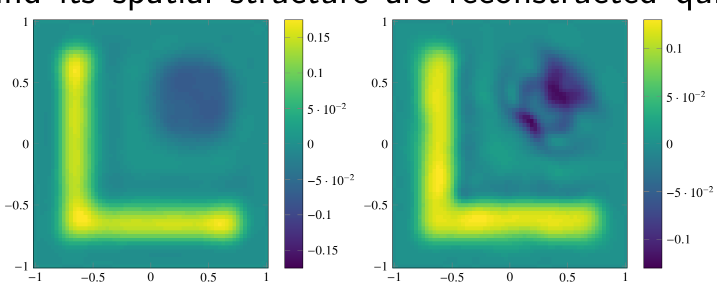
into $F = S \circ P$, with S as in the abstract framework. Since the operator P is easy to analyze, important properties (like differentiability and ill-posedness) of S directly transfer to F. For example, F is differentiable in all parameters $c, \rho \in W^{3,\infty}([0, T]; L^\infty(\Omega))$ that are uniformly positive.

Numerical Results

Numerical experiments for the acoustic wave equation are promising, both in 2D and in 3D.[4]
The figure on the right shows a possible exact parameter. It is discontinuous in the two spatial variables, and its time-dependence is given by a trigonometric polynomial. If we use this parameter as a ground truth for c (assuming a known background value) and consider simulated data with 1% noise then the regularization method REGINN yields a reconstruction with a relative $H^1([0, T]; L^2(\Omega))$-error of $46\,\%$. Nevertheless, both the time-dependence of the parameter and its spatial structure are reconstructed quite well.



Ground truth at $t = \pi/2$



Reconstruction evaluated at $t = \frac{1}{2}\pi$ (left) and $t = \frac{4}{3}\pi$ (right), 1% noise

Furthermore, the reconstruction error seems to converge to zero for vanishing noise level; e.g. for a noise level of 0.01% it decreases to about $12\,\%$.

[1] T. Gerken und A. Lechleiter. "Reconstruction of a Time-Dependent Potential from Wave Measurements". In: Inverse Problems 33.9 (2017), S. 094001
[2] T. Gerken und S. Grützner. "Dynamic Inverse Wave Problems – Part I: Regularity for the Direct Problem". In: Inverse Problems (2019)
[3] T. Gerken. "Dynamic Inverse Wave Problems – Part II: Operator Identification and Applications". In: Inverse Problems (2019)
[4] T. Gerken. "Dynamic Inverse Problems for the Acoustic Wave Equation". In: Time-Dependent Problems in Imaging and Parameter Identification. Springer, 2020

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