PhD projects R3: Mathematical Data Analysis

Application of Δ-Complexes in Topological Data Analysis (2nd cohorte)

Discrete Morse Theory in the Topological Data Analysis (2nd cohorte)

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Topological Data Analysis

The idea of topological data analysis can be summarized in the following four steps:

1. Consider data which could be of different types, e.g. metric spaces, networks or digital images, that is assumed to be part of some unknown underlying space.
2. Construct filtered complexes based on the data.
3. Analyse the filtered complex, e.g. by using persistent homology which is usually given in the form of a barcode.
4. Interpret the results.

Application of Δ-Complexes in TDA (Lena Ranke)

In this project, the second step is of interest. Right now, there are several different filtered simplicial complexes that can be used to approximate the structure of the given data, for example the Vietoris-Rips Complex, the Alpha Complex or the Witness Complex.
Desired features of any filtered complex include

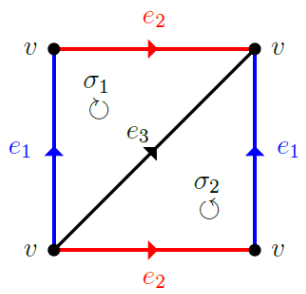
- an appropriate approximation of the unknown underlying space,
- a low number of cells in high dimensions,
- easy computation.

To achieve these, this project's goal is to introduce a filtered complex that consists of Δ-Complexes rather than simplicial ones. [1]

Why use Δ-Complexes?

The Δ-Complex construction allows for a broader variety of complexes. While both are a collection of simplices of different dimensions, the gluing maps of the Δ-Complex have to satisfy less properties than those of simplicial complexes. With this complexes that have loops or double edges for example become possible and that means that an approximation of a space using a Δ-Complex has potentially less cells than an approximation using a simplicial one. Since simplicial complexes are also Δ-Complexes, we get at most the number of cells we would have if we used a simplicial complex in the first place.

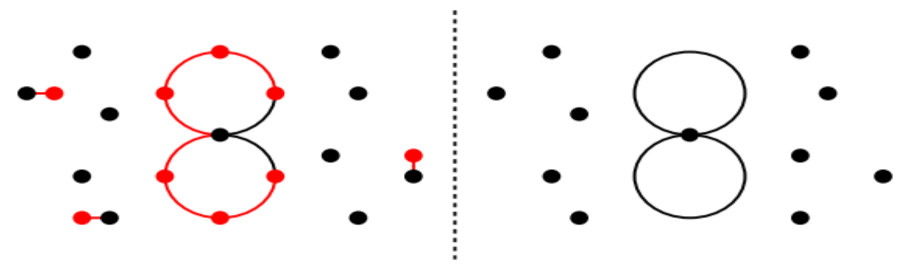
An easy example of this is the torus T^2. If we want to describe it as a simplicial complex, we need at least 7 vertices, 21 edges and 14 triangles. As a Δ-Complex it can be described using only 1 vertex, 3 edges and 2 triangles, as visualized on the right.[2]



The first goal of this project is to consider ways to define a filtered Δ-Complex that can approximate the topology of the original space, the second one is to find an algorithm that can lead to its implementation in application and lastly there should be a discussion if the introduction of Δ-Complexes into this field has had advantages in some areas of TDA over the already used filtered simplicial complexes.

Discrete Morse Theory in TDA (Gideon Klaila)

This project cares about the third step. The calculation of the persistent homology depends on the amount of cells in a filtered complex. To improve the calculation, the project applies Discrete Morse Theory to the complexes to reduce the amount of cells.



Example of benefits of DMT in TDA, non-critical cells in red

Left hand side:

- Filtered complex
- 20 zero-cells
- 11 one-cells

Right hand side:

- Reduced filtered complex
- 11 zero-cells
- 2 one-cells

Basics of Discrete Morse Theory

Based on the Morse Theory founded in 1934 by Marston Morse, Robin Forman created 1998[3] the Discrete Morse Theory to reduce CW complex:

Theorem 1. *Let M be a CW complex and let $m_d(f)$ denote the number of critical cells for a Discrete Morse Function f in dimension d. Then M is homotopy equivalent to a CW complex with exactly $m_d(f)$ cells in dimension d, $0 \leq d \leq \dim M$.*

Dmitry Kozlov expanded the theory[4] by replacing the Discrete Morse Functions with acyclic matchings:

Theorem 2. *Let K be an abstract simplical complex and let $M \subseteq K$ be some set of simplices of K. Given an acyclic matching $\mu : M \to M$, than there exists a CW complex X, such that*

- *for each dimension d, the number of d-cells in X is equal to the number of d-simplices in K, which are critical with respect to μ,*
- *there is a homotopy equivalence $K \simeq X$.*

These theorems can be used to reduce cell complexes without change of homology.

[1] Otter, N., & Porter, M.A., & Tillmann, U., & et al. (2017, August). A roadmap for the computation of persistent homology. In EPJ Data Sci. 6, 17.
[2] Hatcher, A. (2006). Algebraic Topology. Cambridge Univ. Press.
[3] Forman, R. (1998). Morse theory for cell complexes. Advances in Mathematics, 134(1)
[4] Kozlov, D. (to appear). Organized collapse. Graduated Studies in Mathematics, Vol. 300

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